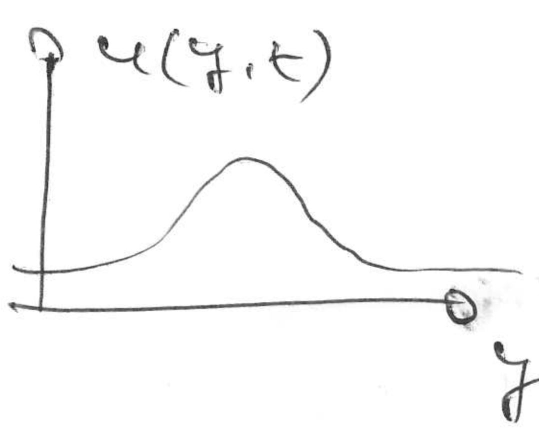
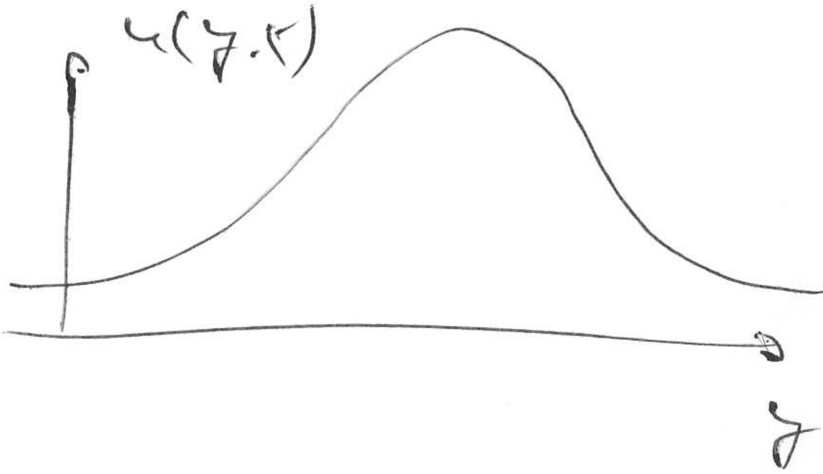
~~Similrito~~

Similarity soln

Often solns are self-similar, i.e. they have the same "shape" but possibly a different scale at different times.

E.g.:





Early time Late time

The generic form of such solns is

$$u(y,t) = \underbrace{a(t)}_{\text{This scales the amplitude}} f\left(\frac{y}{b(t)}\right)$$

This scales the amplitude

"$b(t)$ scales the "width" of f.

Sim. solns don't always exist.

Try it!

- Often sim. solns reduce PDEs to ODEs for $f(\eta)$ where $\eta = \frac{y}{b(t)}$
- The existence of sim. solns is often motivated by dimensional arguments.

<u>Example:</u>

Rayleigh's jerked plate.

U at $t > 0$

parallel flow: $\underline{u} = u(y,t)\,\underline{e}_x$

Govn. eqns:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

IC: $u = 0$ at $t = 0$

BC: $u|_{y=0} = U$ for $t > 0$

$u \to 0$ as $y \to \infty$.

- PDE is linear!
- All eqns are homogeneous apart from 1st BC.

$$u(y, t; \nu, U) = U f(\underbrace{y, t; \nu})$$

(assuming uniqueness)

$\Rightarrow$ f must not depend on U!

Check dimensions.

dimension of u is m/sec [4]
& the same as those of U.

$\Longrightarrow$ $f(y, t; \nu)$ must be a dimensionless fct. of its arguments.

so y, t & ν must combine to something dimensionless. $f(y, t; \nu)$

$[y] = m$

$[t] = sec$

$$[\nu] = \frac{\left[\frac{\partial u}{\partial t}\right]}{\left[\frac{\partial^2 u}{\partial y^2}\right]} = \frac{\cancel{m}}{sec^{\cancel{2}}} \frac{\cancel{sec}\, m^2}{\cancel{m}}$$

$$[\nu] = \frac{m^2}{sec}$$

Dimensionless combination is

$$\eta = \frac{\nu t}{y^2} \quad \text{or} \quad \eta = \frac{y}{\sqrt{\nu t}}$$

The latter choice is the only one that gives the required similarity form

$$u(y,t;\nu,U) = U f(y,t;\nu)$$

$$= U f\Big(\underbrace{\frac{y}{\sqrt{\nu t}}}\Big)$$

[compare to

$$u(y,t) = a(t)\, f\Big(\overbrace{\frac{y}{b(t)}}^{\eta}\Big)$$
]

One tends to try to keep the sim. variable η linear in y.

"Ansatz"

$$u(y,t;\nu,U) = U f\Big(\underbrace{y(\nu t)^{-1/2}}\Big)$$

into PDE

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad \eta = y(\nu t)^{-1/2}$$

$$\frac{\partial u}{\partial t} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial t}$$

$$\boxed{\frac{\partial u}{\partial t} = U f' \, y\left(-\tfrac{1}{2}\right)\nu^{-1/2}\, t^{-3/2}}$$

$$\frac{\partial u}{\partial y} = U \frac{df}{d\eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial y} = U f' \frac{1}{\sqrt{\nu t}}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = U f'' \frac{1}{\nu t}}$$

into PDE:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$-\frac{1}{2}\cancel{U} \underbrace{\frac{y}{\sqrt{\nu t}}}_{\eta} \cancel{\frac{1}{t}} f' = \cancel{\nu}\cancel{U} f'' \cancel{\frac{1}{\nu t}}$$

$$\boxed{f'' + \frac{1}{2}\eta f' = 0}$$ an ODE!

BC: $u(y,t) = U f\left(\frac{y}{\sqrt{\nu t}}\right)$ 7

$u = U$ at $y = 0$

$U = U f(0)$

$$\boxed{f(0) = 1}$$

$u \to 0$ as $y \to \infty$

$U f(\eta) \to 0$ as $\eta \to \infty$

$$\boxed{f(\eta) \to 0 \text{ as } \eta \to \infty}$$

IC: $u = 0$ when $t = 0$

$u \to 0$ as $t \to 0$

$$\boxed{f(\eta) \to 0 \text{ as } \eta \to \infty}$$

again !!

2 cond for a 2^{nd} order ODE!