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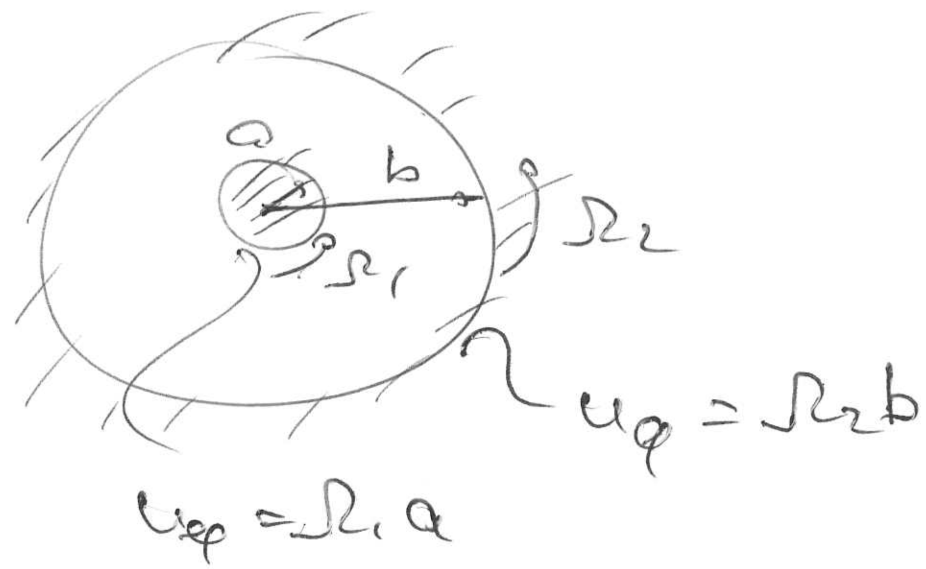


Assume: $\underline{u} = u_\varphi(r)\,\underline{e_\varphi} = v(r)\,\underline{e_\varphi}$

$$\nabla p = \underline{0}$$

Conti & z-mom. ✓

φ-mom: $\Longrightarrow v(r) = Ar + \frac{B}{r}$

A & B from BC.

~~$v(b) = \Omega_2 b$~~

$$\boxed{\begin{aligned} v(a) &= \Omega_1 a = Aa + \frac{B}{a} \\ v(b) &= \Omega_2 b = Ab + \frac{B}{b} \end{aligned}}$$

$u = w = 0$ $\quad$ $\nabla p = 0$ $1\frac{1}{2}$

wrong

$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial z} = \frac{\partial}{\partial t} = 0 \; ; \; v(r)$!

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial u}{\partial \varphi}} + \cancel{w\frac{\partial u}{\partial z}} - \frac{v^2}{r} = \cancel{-\frac{1}{\rho}\frac{\partial P}{\partial r}} + \nu\left[\cancel{\nabla^2 u} - \cancel{\frac{u}{r^2}} - \cancel{\frac{2}{r^2}\frac{\partial v}{\partial \varphi}}\right],$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \boxed{\nu\left[\nabla^2 v - \frac{v}{r^2}\right.} \left.+ \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right],$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0.$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

$$v(r) = \frac{1}{b^2 - a^2} \left\{ (b^2 \Omega_2 - a^2 \Omega_1) r - \frac{a^2 b^2 (\Omega_2 - \Omega_1)}{r} \right\}$$

<u>TEST:</u> if $\Omega_1 = \Omega_2 = \Omega$

$v(r) = \Omega r$ rigid body rotation ✓

But contradiction with radial mom. eqn.

$$\cancel{-\frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \mu [\quad]}$$

wrong / inconsistent.

Fix by allowing $p = p(r)$ (centrifugal forces)

Luckily all other eqns remain the same

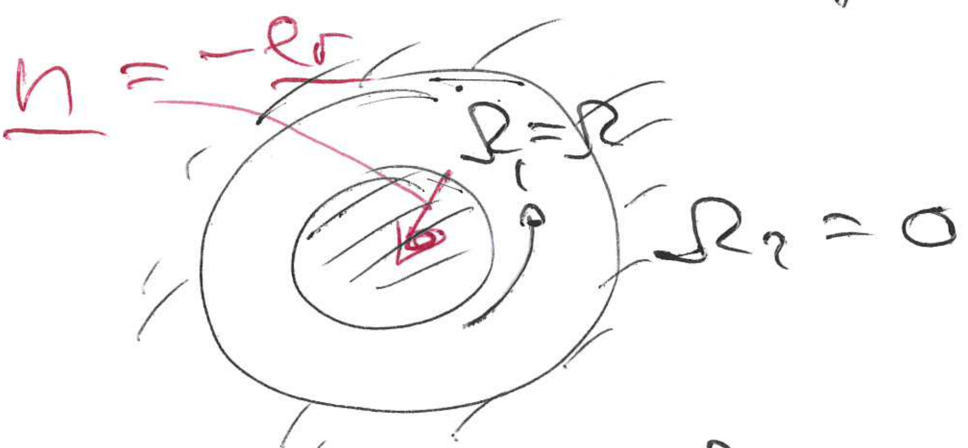
⇒ determine pressure from

$$\frac{\partial p}{\partial r} = \frac{dp}{dr} = \rho \frac{v^2(r)}{r} \leftarrow \text{known}$$

Integrate to obtain $p(r)$.
(omitted)

Note: integration introduces an arbitrary constant into pressure. This is typical for enclosed, incompressible fluids.

Traction on rotating shaft:



$$v(r) = \frac{a^2 \Omega}{b^2 - a^2}\left(\frac{b^2}{r} - r\right)$$

Traction on ~~shaft~~ fluid $\underline{t}$ (4-

$$\underline{t} = \underset{\sim}{T} \cdot \underline{n}$$

or $t_i = \tau_{ij} n_j \quad i,j = r, \varphi, z$

$$\tau_{ij} = -p\,\delta_{ij} + 2\mu \underbrace{e_{ij}}_{\text{rate of strain tensor from formula sheet.}}$$

$$\underline{n} = -\underline{e_r} = n_r \underline{e_r} + n_\varphi \underline{e_\varphi} + n_z \underline{e_z}$$

$$n_r = -1 \quad n_\varphi = n_z = 0$$

only $e_{r\varphi} = \frac{1}{2} r \frac{\partial}{\partial r}\left(\frac{v}{r}\right)$ known. is nonzero

$$e_{r\varphi} = -\frac{a^2 b^2 \Omega}{b^2 - a^2} \frac{1}{r^2}$$

$$e_{r\varphi}\Big|_{r=a} = -\frac{b^2 \Omega}{b^2 - a^2}$$

$u = u_r = 0$; $w = u_z = 0$ (4½

$v = u_\varphi = v(r)$

$$\cancel{\epsilon_{rr} = \frac{\partial u}{\partial r}} \qquad \epsilon_{\varphi\varphi} = \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{u}{r}}$$

$$\cancel{\epsilon_{zz} = \frac{\partial w}{\partial z}} \qquad \boxed{\epsilon_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right.} + \left.\cancel{\frac{1}{r}\frac{\partial u}{\partial \varphi}}\right]$$

$$\epsilon_{\varphi z} = \frac{1}{2}\left[\cancel{\frac{1}{r}\frac{\partial w}{\partial \varphi}} + \cancel{\frac{\partial v}{\partial z}}\right] \qquad \epsilon_{rz} = \frac{1}{2}\left[\cancel{\frac{\partial u}{\partial z}} + \cancel{\frac{\partial w}{\partial r}}\right]$$

$$t_i = -p n_i + 2\mu e_{ij} n_j$$

$i = r$:

$$t_r = -p \underbrace{n_r}_{-1} + 2\mu \left[\cancel{e_{rr} n_r} + \cancel{e_{r\varphi} n_\varphi} + \cancel{e_{rz} n_z} \right]$$

$$t_r = p \ (r = a)$$

$i = \varphi$:

$$t_\varphi = -p \cancel{n_\varphi} + 2\mu \left[e_{\varphi r} \underbrace{n_r}_{-1} + \cancel{e_{\varphi\varphi} n_\varphi} + \cancel{e_{\varphi z} n_z} \right]$$

$$t_\varphi = -2\mu e_{\varphi r} = +2\mu \frac{b^2 \Omega}{b^2 - a^2}$$

Also:

$$t_z = 0$$

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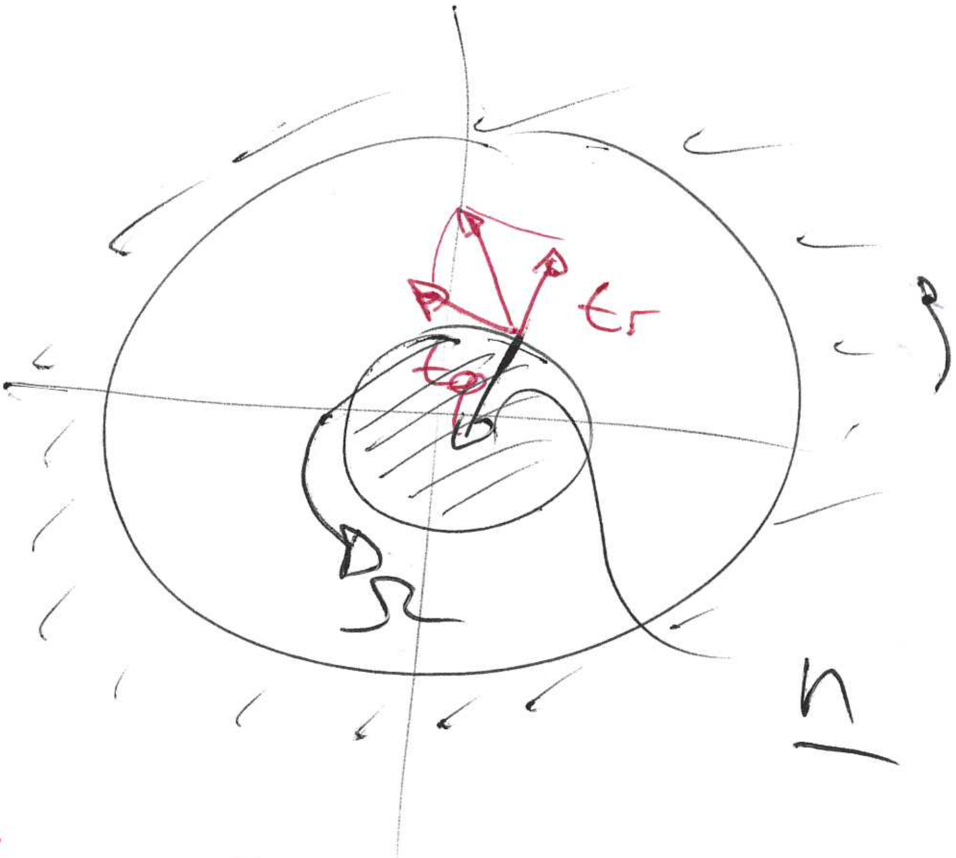

$t_r = p$

$t_\varphi = \ldots > 0$

$$t_\varphi = 2\mu \frac{b^2 \Omega}{b^2 - a^2}$$

Note: Had assumed a very simple flow field & found a soln.

$\Rightarrow$ Q: Is this the only one?

No (Tuesday)