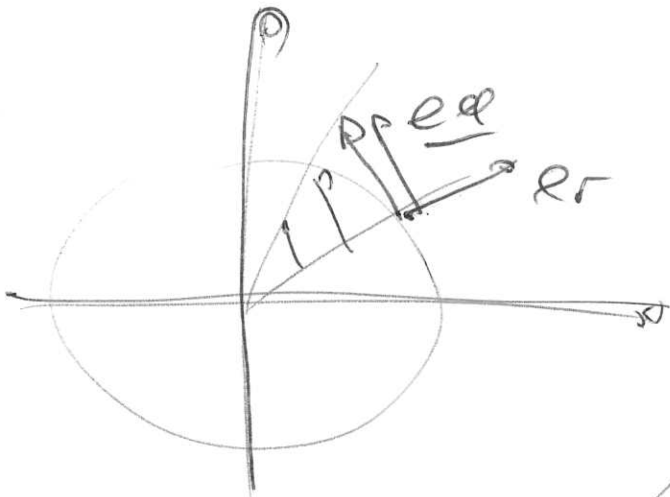
Rigid body rotation

$$\underline{u} = \Omega r \, \underline{e_\varphi} = u_r \underline{e_r} + u_\varphi \underline{e_\varphi} + u_z \underline{e_z}$$



$u_r = 0 = u$

$u_\varphi = \Omega r = v$

$u_z = 0 = w$

only nonzero comp. of NS:

r-component: $+\rho \frac{v^2}{r} = + \frac{\partial p}{\partial r}$

$$\rho \frac{\Omega^2 r^{\cancel{2}}}{\cancel{r}} = \frac{\partial p}{\partial r}$$

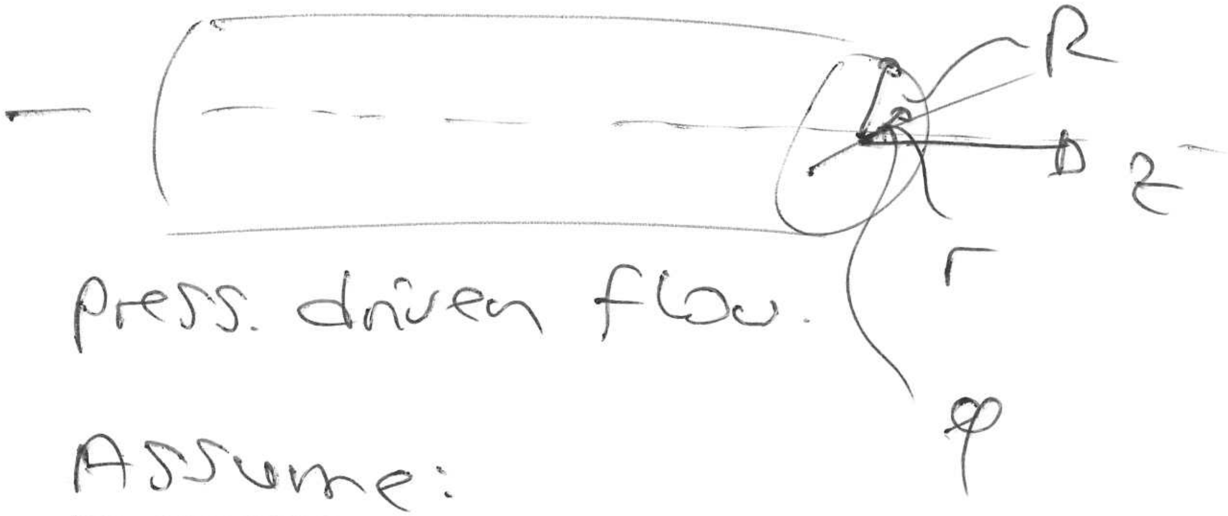
$$p = p_0 + \frac{1}{2} \rho \Omega^2 r^2$$

p_0: const.

centrifugal forces!

Example:

Hagen - Poiseuille flow in a circular pipe



press. driven flow.

Assume:

- $\underline{u}$ indep. of z & φ
- $\underline{u}$ unidirectional & steady

$\Rightarrow \underline{u} = u_z \underline{e_z} = u_z(r)\, \underline{e_z} = w(r)\, \underline{e_z}$

- $p = p(r, z)$

Into N-St.

r-comp.: $\quad 0 = -\frac{1}{\rho} \frac{\partial p}{\partial r} \Rightarrow p = p(z)$

φ-comp: $\quad 0 = 0$

z - comp:

$w = w(r)$
$p = p(z)$

$$\cancel{w\frac{\partial w}{\partial z}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{fct. of } z} + \underbrace{\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)}_{\text{fct of } r}$$

$\Rightarrow \frac{\partial p}{\partial z}$ must be indep. of z

$$\frac{\partial p}{\partial z} = G \quad \text{a constant.}$$

$$\frac{G}{\mu} = \left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)$$

$$= \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$$

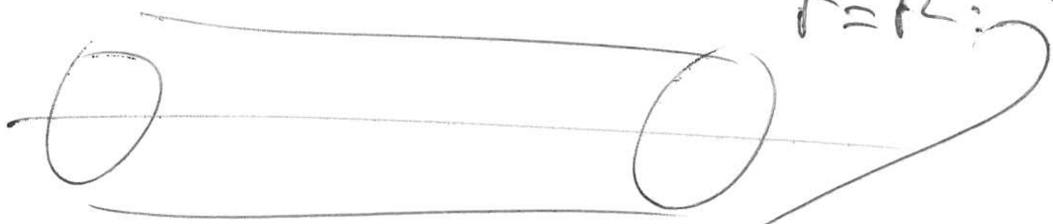
$$\frac{1}{2}\frac{G}{\mu}r^2 + A = r\frac{dw}{dr}$$

$$\frac{dw}{dr} = \frac{1}{2}\frac{G}{\mu}r + \frac{A}{r}$$

$$w(r) = \frac{1}{4}\frac{G}{\mu}r^2 + A\ln r + B$$

const.

2 BC:



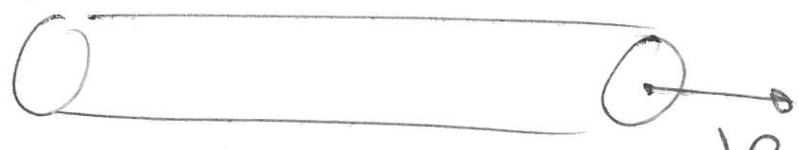
$w(r=R) = 0$
(no slip)

soln has to remain finite at $r = 0$. $\Rightarrow A = 0$.

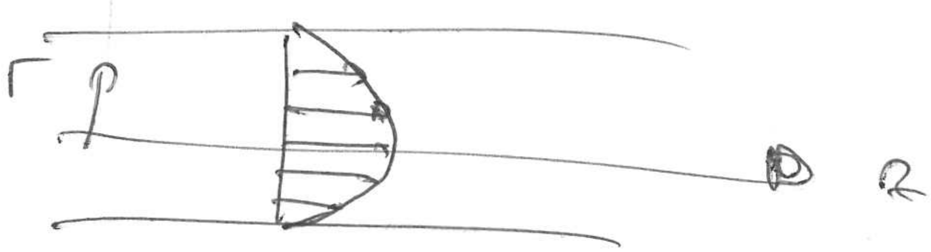
$$\Rightarrow w(r) = \frac{1}{4}\frac{G}{\mu}\underbrace{(r^2 - R^2)}_{\text{neg in pipe}} \geq 0$$

$$G = \frac{\partial p}{\partial z} \Rightarrow p = p_0 + Gz$$

p_0 const.

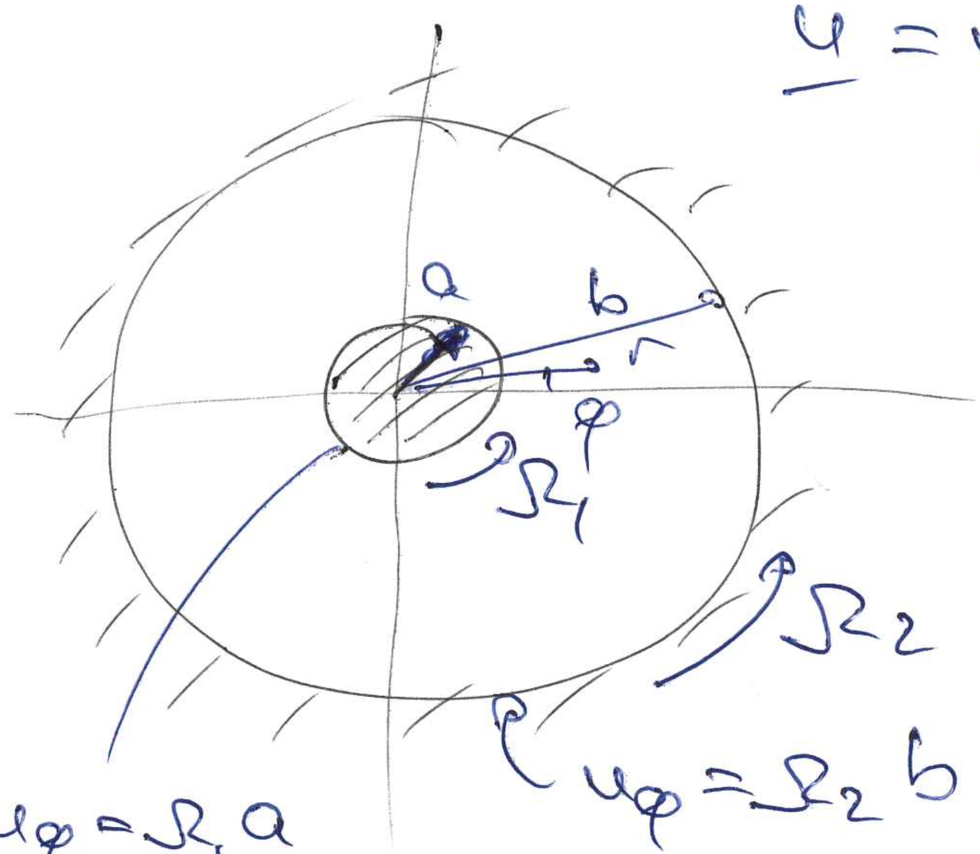


$$\frac{\partial p}{\partial z} = G < 0$$



Example Circular Couette flow

$\underline{u} = u_r \underline{e_r} + u_\varphi \underline{e_\varphi} + u_z \underline{e_z}$



Assumptions

- steady
- $\underline{u} = u_\varphi \underline{e_\varphi}$
- $\frac{\partial}{\partial \varphi} = 0$
- $\frac{\partial}{\partial z} = 0$

$\Big\}\ \underline{u} = u_\varphi(r)\underline{e_\varphi}$

- As in planar Couette: flow driven by wall & $\nabla p = \underline{0}$

$u = 0$
$v = v(r)$
$w = 0$

$\nabla p = 0$

$(5\tfrac{1}{2})$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \varphi} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right],$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial v}{\partial \varphi}} + \cancel{w\frac{\partial v}{\partial z}} + \cancel{\frac{uv}{r}} = \cancel{-\frac{1}{\rho r}\frac{\partial P}{\partial \varphi}} + \boxed{\nu\Big[\nabla^2 v - \frac{v}{r^2}} \cancel{+\frac{2}{r^2}\frac{\partial u}{\partial \varphi}}\Big],$$

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial r}} + \cancel{\frac{v}{r}\frac{\partial w}{\partial \varphi}} + \cancel{w\frac{\partial w}{\partial z}} = \cancel{-\frac{1}{\rho}\frac{\partial P}{\partial z}} + \cancel{\nu\nabla^2 w},$$

$$\operatorname{div}\mathbf{u} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(ru)} + \cancel{\frac{1}{r}\frac{\partial v}{\partial \varphi}} + \cancel{\frac{\partial w}{\partial z}} = 0.$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}} + \cancel{\frac{\partial^2}{\partial z^2}}.$$

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φ - comp:

$$\nabla^2 u - \frac{u}{r^2} = 0$$

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)} - \frac{u}{r^2} = 0$$

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

$$\boxed{r^2\frac{d^2 u}{dr^2} + r\frac{du}{dr} - u = 0}$$

Euler ODE

Ansatz: $u \sim r^\lambda$

$$r^2 \lambda(\lambda-1) r^{\lambda-2} + r\lambda r^{\lambda-1} - r^\lambda = 0$$

$$r^\lambda \underbrace{\left(\lambda(\lambda-1) + \lambda - 1\right)}_{=0} = 0$$

$$\lambda^2 \cancel{-\lambda} \cancel{+\lambda} - 1 = 0$$

$\lambda = \pm 1$

$\Rightarrow u(r) = A r + \frac{B}{r}$

const.