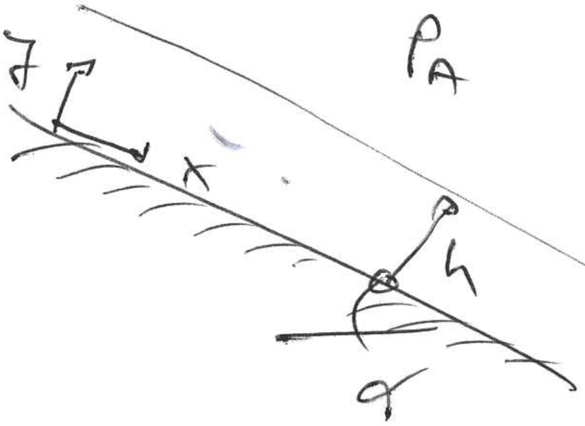


$$|\underline{g}| = g$$

$$\underline{f} = f_x \underline{e_x} + f_y \underline{e_y}$$

$$f_x = g \sin\alpha \qquad f_y = -g\cos\alpha.$$

Assume:
- parallel flow in x-dir
- steady
- indep. of z

$$\cancel{\rho \frac{\partial u}{\partial t}} = -\frac{\partial p}{\partial x} + \underbrace{\rho g \sin\alpha}_{f_x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha} \quad (1)$$

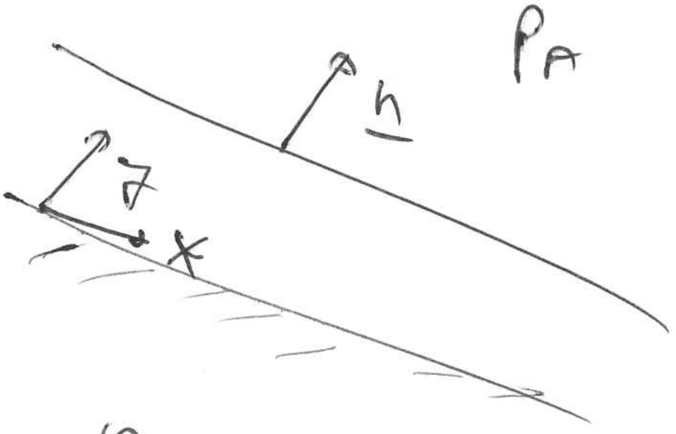
y - comp:

$$\boxed{0 = -\frac{\partial p}{\partial y} \underbrace{- \rho g \cos\alpha}_{f_y}} \quad (2)$$

BC: $u(y=0)=0$ (no slip) (2

At $y=h$: free surface; fluid meets (inviscid) air of a pressure p_A.

Surface of fluid is subject to an applied traction from the air



$$\underline{n} = n_x \underline{e_x} + n_y \underline{e_y} = \underline{e_y}$$

$$n_x = 0 \qquad n_y = 1$$

Traction onto fluid

$$\underline{t} = -p_A \underbrace{\underline{n}}_{\underline{e_y}} = t_x \underline{e_x} + t_y \underline{e_y}$$

$$t_x = 0 \qquad t_y = -p_A$$

In index notation:

$$n_1 = 0 \qquad n_2 = 1$$

$$t_1 = 0 \qquad t_2 = -p_A$$

<u>Traction BC:</u>

$$t_i = \tau_{ij} n_j \quad \text{at } y = x_2 = h$$

$$t_i = \left[-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] n_j$$

$$t_i = -p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

For all 2 indices $i = 1, 2$

<u>$i = 2$:</u>

$$\underbrace{t_2}_{-p_A} = -p\underbrace{n_2}_{1} + \mu\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)\cancel{n_1} + \mu\left(\cancel{\frac{\partial u_2}{\partial x_2}} + \cancel{\frac{\partial u_2}{\partial x_2}}\right) n_2$$

$n_1 = 0$

$u_2 = 0$ (parallel flow)

$$\boxed{p = p_A \quad \text{at } x_2 = y = h}$$

i=1:

$$\underset{0}{\underbrace{t_1}} = -p\,\overset{0}{\cancel{n_1}} + \mu\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\right)\overset{0}{\cancel{n_1}}$$

$$+\mu\left(\frac{\partial u_1}{\partial x_2}+\underset{u_2=0}{\cancel{\frac{\partial u_2}{\partial x_1}}}\right)\underset{1}{\underbrace{n_2}}$$

$$\boxed{0 = \mu\frac{\partial u_1}{\partial x_2} = \mu\frac{\partial u}{\partial y} \quad \text{at } y=h}$$

Integrate (2) w.r.t y:

$$\frac{\partial p}{\partial y} = -\rho g\cos\alpha$$

$$p(x,y) = -\rho g\cos\alpha\, y + f(x)$$

Apply press. BC:

$$\underline{p(y=h) = p_A = -\rho g\cos\alpha\, h + \underline{f(x)}}$$

$$\boxed{p(x,y) = p_A + \rho g\cos\alpha\,(h-y)}$$

Note: - press. increases throu
thickness of film
(hydrostatics)

- press. does not depend on x: flow not driven by press. but by gravity.

Now (1)

$$0 = -\cancel{\frac{\partial p}{\partial x}} + \rho g \sin\alpha + \mu \frac{\partial^2 u}{\partial y^2}$$

integrate twice w.r.t. y:

$$u = -\frac{1}{2}\frac{\rho g \sin\alpha}{\mu} y^2 + Ay + B$$

ν

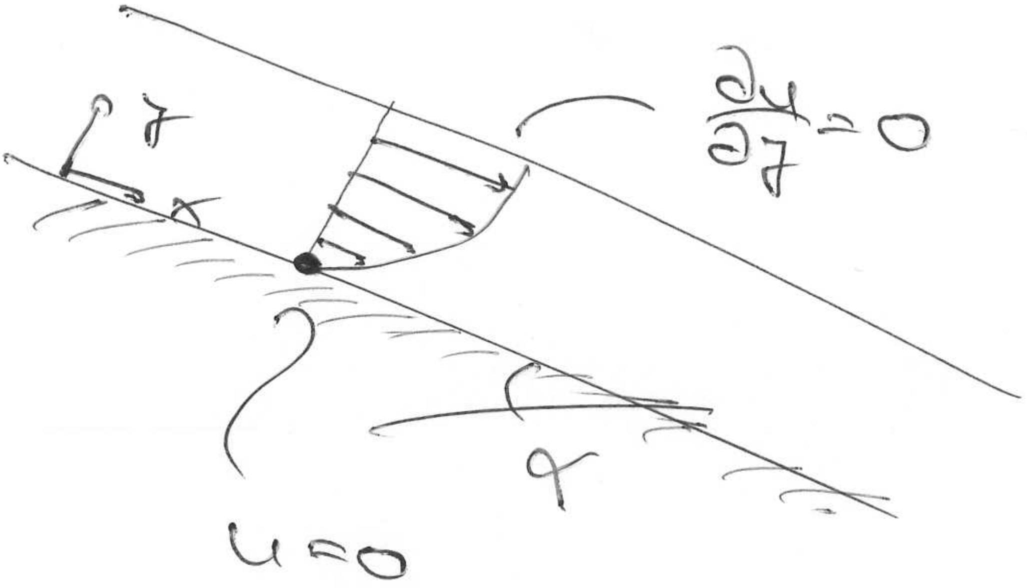
const.

Apply BC:

$$u(y=0) = 0$$

$$\mu \frac{\partial u}{\partial y}\Big|_{y=h} = 0$$

$$u(x,y) = \frac{g\sin\alpha}{\nu}\left[hy - \frac{1}{2}y^2\right]$$



Example: The vibrating plate

Assume:
- parallel flow
- flow driven by wall
  $\Rightarrow$ no need for press. gradient. $\frac{dp}{dx} = G = 0$
- no body force.

$$\rho \frac{\partial u}{\partial t} = \cancel{-\frac{\partial p}{\partial x}} + \cancel{\rho f_x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}} \right)$$

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$