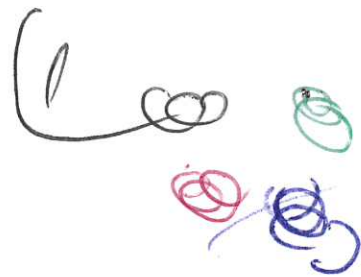


Index notation



Various ways of writing a vector:

$$\begin{array}{l} \underline{a} \\ \uparrow \\ \text{symbolic} \end{array} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \begin{array}{l} \text{in components} \\ \text{rel. to some} \\ \text{basis, here} \\ (\underline{e}_1, \underline{e}_2, \underline{e}_3) = \\ (\underline{i}, \underline{j}, \underline{k}) \end{array} \quad \begin{array}{l} \\ \\ \\ \\ \end{array} \quad \begin{array}{l} \\ \\ \\ \\ \text{(components)} \end{array}$$

Convention 1: Simply write down one generic term of each vector (egh).

~~c~~

$$\underline{c} = \underline{b} + \underline{a} \Rightarrow c_i = b_i + a_i$$

i is a "free index"
& takes values 1, 2, 3

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} 0_1 \\ 0_2 \\ 0_3 \end{pmatrix}.$$

(2)

Ex:

$$\nabla \phi = \frac{\partial \phi}{\partial x_1} \underline{e}_1 + \frac{\partial \phi}{\partial x_2} \underline{e}_2 + \frac{\partial \phi}{\partial x_3} \underline{e}_3$$

$$= \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix} \Rightarrow \frac{\partial \phi}{\partial x_i}.$$

Convention 2: Summation convention

Rule: Automatically sum over repeated indices (dummy indices)

Example:

$$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$= \sum_{i=1}^3 u_i v_i = u_i v_i$$

"Dummy index" because their name is irrelevant (3)

$$\underline{u} \cdot \underline{v} = u_i v_i = u_k v_k$$

Example:

$$\begin{aligned} \text{div } \underline{u} &= \nabla \cdot \underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\ &= \frac{\partial u_i}{\partial x_i} = \frac{\partial u_k}{\partial x_k} = \frac{\partial u_j}{\partial x_j} = \dots \end{aligned}$$

Higher order "tensors"

So far: Index notation represents vectors: 3 components represented by one free index.

Higher order tensors in many contexts: E.g.

$$\underline{\sigma} = \underline{T} \cdot \underline{n}$$

stress vector. stress tensor (≈ matrix) outer unit normal vector.

Matrix-vector product:

(4)

$$v_i = T_{ij} n_j$$

↑
one
free
index: i

one free
index: j

Note: All terms in any
eqn must have the same
free indices.

A special 2nd order tensor
is the Kronecker Delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

(unit matrix)

$$[\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

δ_{ij} has an interesting property when used in summations:

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$$b_j = \sum_i a_i \delta_{ij} = a_j$$

(cancels out)

" δ_{ij} exchanges indices"

Even higher order tensors

appear too.

E.g.:

$$\sigma_{ij} = \epsilon_{ijke} \epsilon_{ke}$$

stress tensor (2nd order) 4th order tensor strain tensor (2nd order)

Disclaimer:

(6)

Not every "thing" with
subscripts is a tensor!
