Index notation

Various ways of writing a vector:

$$\underline{a} = a_1 \underline{e_1} + a_2 \underline{e_2} + a_3 \underline{e_3} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

symbolic | in components rel. to some basis, here $(\underline{e_1}, \underline{e_2}, \underline{e_3}) = (\underline{i}, \underline{j}, \underline{k})$ | (components)

Convention 1: Simply write down one generic term of each vector (eqn).

$$\underline{c} = \underline{b} + \underline{a} \Rightarrow c_i = b_i + a_i$$

i is a "free index" & takes values 1, 2, 3

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

<u>Ex:</u>

$$\nabla \phi = \frac{\partial \phi}{\partial x_1} \underline{e}_1 + \frac{\partial \phi}{\partial x_2} \underline{e}_2 + \frac{\partial \phi}{\partial x_3} \underline{e}_3$$

$$= \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix} \implies \frac{\partial \phi}{\partial x_i}.$$

<u>Convention 2: Summation convention</u>

<u>Rule:</u> Automatically sum over repeated indices (dummy indices)

<u>Example:</u>

$$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$= \cancel{\sum_{i=1}^{3}} u_i v_i = u_i v_i$$

"Dummy index" because their name is irrelevant

$$\underline{u} \cdot \underline{v} = \underbrace{u_i v_i} = \underbrace{u_k v_k}$$

Example:

$$\operatorname{div} \underline{u} = \nabla \cdot \underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \frac{\partial u_i}{\partial x_i} = \frac{\partial u_k}{\partial x_k} = \frac{\partial u_j}{\partial x_j} = \ldots$$

Higher order "tensors"

So far: Index notation represents vectors: 3 components represented by one free index.

Higher order tensors arise in many contexts: E.g.

$$\underline{\sigma} = \underset{\sim}{T} \cdot \underline{n}$$

stress vector. — stress tensor ($\approx$ matrix) — outer unit normal vector.

Matrix-vector product:

$$\sigma_i = T_{ij} n_j$$

one free index: i

one free index: i

Note: All terms in any eqn must have the same free indices.

A special 2nd order tensor is the Kronecker Delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

(unit matrix)

$$[\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

δ_{ij} has an interesting property when used in summations:

$$b_j = \boxed{a_i \delta_{ij}} = \sum_{i=1}^{3} a_i \delta_{ij} = \boxed{a_j}$$

(write out)

"δ_{ij} exchanges indices"

Even higher order tensors

appear too.

E.g.:

$$\sigma_{ij} = E_{ijkl} \, \varepsilon_{kl}$$

σ_{ij}: stress tensor (matrix)

E_{ijkl}: 4th order tensor.

ε_{kl}: strain tensor (matrix)

Disclaimer:

Not every "thing" with subscripts is a tensor!