Chapter 4

Boundary and initial conditions

4.1 Initial conditions

- For time-dependent problems, an initial condition for the velocity field, i.e. $u_i(x_j, t = 0)$ has to be specified.

4.2 Boundary conditions

- Fig. 4.1 shows a selection of common boundary conditions for flow problems.

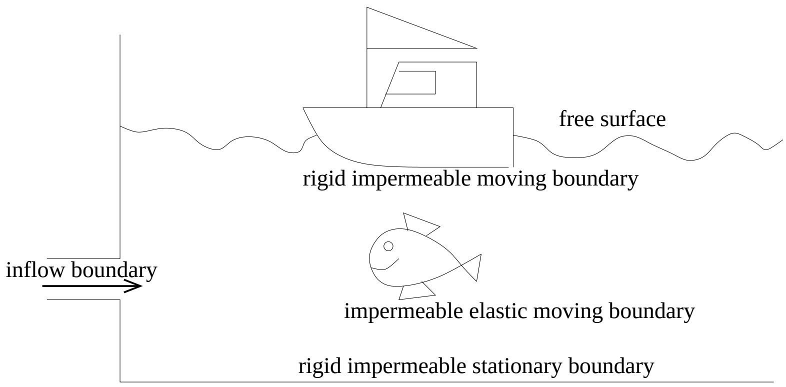

Figure 4.1: Common boundary conditions.

4.2.1 Inflow/outflow boundary conditions

- In many applications, we are only interested in the behviour of the fluid in a small region (for instance, if we want to study the ventilation in a room, it would be impractical to include the earth's entire atmosphere into the model. We would only model the room and treat its interaction with the 'rest of the world' via inflow boundary conditions – e.g. by prescribing the wind velocity through an open window). Hence at inflow (or outflow) boundaries we prescribe the velocity, i.e.

$$u_i = v_i, \tag{4.1}$$

where v_i is a prescribed function.

4.2.2 Solid surfaces

- Most solid surfaces are impermeable to fluid and the fluid 'sticks' to their surfaces. Hence, there is no slip and no penetration, and the fluid particles on the wall move with the velocity of the wall:

$$u_i = w_i, \tag{4.2}$$

where w_i is the (known) velocity of the impermeable wall.

- In the special case where the walls are stationary we have

$$u_i = 0. \tag{4.3}$$

4.2.3 Free surfaces

- Free surfaces occur at the interface between two fluids. Such interfaces require two boundary conditions to be applied: (i) A kinematic condition which relates the motion of the free interface to the fluid velocities at the free surface and (ii) a dynamic condition which is concerned with the force balance at the free surface.

(i) The kinematic boundary condition

– The position of a free surface can always be given in implicit form as $F(x_j, t) = 0$. For instance, in Fig. 4.2 the height of the free surface above the x-axis is specified as $y = h(x, t)$ and an appropriate function $F(x, y, t)$ would be given by $F(x, y, t) = h(x, t) - y$.

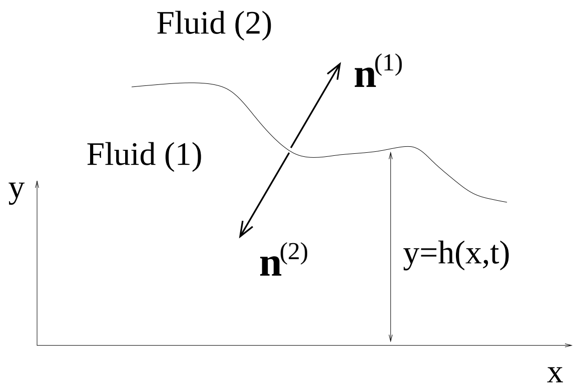

Figure 4.2: Sketch illustrating the conditions at a free surface formed by the interface between two fluids.

– Fluid particles on the free surface always remain part of the free surface, therefore we must have

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u_k \frac{\partial F}{\partial x_k}. \tag{4.4}$$

This is the kinematic boundary condition.

– For surfaces whose position is described in the form $z = h(x, y, t)$, the kinematic boundary condition becomes

$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}, \tag{4.5}$$

where u, v, w are the velocities in the x, y, z directions, respectively.

– For steady problems, we have $\partial F/\partial t = 0$ and the kinematic boundary condition can be written as

$$u_i n_i = 0 \quad \text{or symbolically} \quad \mathbf{u} \cdot \mathbf{n} = 0, \tag{4.6}$$

where $\mathbf{n}$ is the outer unit normal on the free surface. This condition implies that there is no flow through the free surface (but there can be a flow tangential to it!).

(ii) The dynamic boundary condition

– The dynamic boundary condition requires the stress to be continuous across the free surface which separates the two fluids (air and water in Fig. 4.1). The traction exerted by fluid (1) onto fluid (2) is equal and opposite to the traction exerted by fluid (2) on fluid (1). Therefore we must have $\mathbf{t}^{(1)} = -\mathbf{t}^{(2)}$. Since $\mathbf{n}^{(1)} = -\mathbf{n}^{(2)}$ (see Fig. 4.2) we obtain the dynamic boundary condition

$$\tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j, \tag{4.7}$$

where we can use either $\mathbf{n}^{(1)}$ or $\mathbf{n}^{(2)}$ as the unit normal.

– On curved surfaces, surface tension can create a pressure jump across the free surface. The surface tension induced pressure jump is given by

$$\Delta p = \sigma \kappa. \tag{4.8}$$

In this expression σ is the surface tension of the fluid and κ is equal to twice the mean curvature of the free surface, i.e.

$$\kappa = \frac{1}{R_1} + \frac{1}{R_2}, \tag{4.9}$$

where R_1 and R_2 are the principal radii of curvature of the surface (for instance, $\kappa = 2/a$ for a spherical drop of radius a and $\kappa = 1/a$ for a circular jet of radius a). Surface tension acts like a tensioned membrane at the free surface and tries to minimise the surface area. Hence the pressure inside a spherical drop (or inside a circular liquid jet) tends to be higher than the pressure in the surrounding medium.

– If surface tension is important, the dynamic boundary condition has to be modified to

$$\tau_{ij}^{(1)} n_j^{(1)} + \sigma \kappa n_i^{(1)} = \tau_{ij}^{(2)} n_j^{(1)}, \tag{4.10}$$

where $\kappa > 0$ if the centres of curvature lie inside fluid (1).

4.2.4 Other boundary conditions

- Other boundary conditions can occur in special applications. For instance, the presence of an elastic boundary leads to fluid-structure interaction problems in which the fluid velocity has to be equal to the velocity of the elastic wall, while the elastic wall deforms in response to the traction that the fluid exerts on it. At porous walls, the no-penetration condition no longer holds: the volume flux into the wall is often proportional to the pressure gradient at the porous surface. Non-uniformly distributed surfactants (substances which reduce the surface tension) can induce tangential stresses at free surfaces, etc.

4.3 Further remarks

- For an incompressible fluid, the boundary conditions need to fulfill the overall consistency condition

$$\oint_{\partial V} u_i n_i \, dS = 0, \tag{4.11}$$

where ∂V is the surface of the spatially fixed volume in which the equations are solved.

- If there are no free surfaces (and associated dynamic boundary conditions), the pressure is only defined up to an arbitary constant as only the pressure gradient (but not the pressure itself) appears in the Navier-Stokes equations.

- For initial value problems, the initial velocity field (at $t = 0$) already has to fulfill the incompressibility constraint.

These remarks are particularly important for the numerical solution of the Navier-Stokes equations.