EXAMPLE SHEET IX

1) Scales:

$u = U\tilde{u}$ $\qquad$ $x = L\tilde{x}$

$v = V\tilde{v} = \frac{h_0}{L} U \tilde{v}$ $\qquad$ $y = h_0 \tilde{y}$

$p = P\tilde{p} = \frac{\mu U}{h_0} \frac{1}{(h_0/L)}$ $\qquad$ $t = \frac{L}{U}\tilde{t}$

[Remember: The scale for v was derived from the requirement that the terms in the continuity equation balance each other; the pressure scale followed from the requirement that the pressure gradient must balance the viscous dissipation]

Into y-N.St.

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho\left(\frac{\left(\frac{h_0}{L}U\right)}{(L/U)}\frac{\partial\tilde v}{\partial\tilde t} + U\frac{\left(\frac{h_0}{L}U\right)}{L}\tilde u\frac{\partial\tilde v}{\partial\tilde x} + \frac{h_0}{L}U\frac{\left(\frac{h_0}{L}U\right)}{h_0}\tilde v\frac{\partial\tilde v}{\partial\tilde y}\right)$$

$$= -\frac{\left(\frac{\mu U}{h_0}\right)}{(h_0/L)\,h_0}\frac{\partial\tilde p}{\partial\tilde y} + \mu\left(\frac{\left(\frac{h_0}{L}U\right)}{L^2}\frac{\partial^2\tilde v}{\partial\tilde x^2} + \frac{\left(\frac{h_0}{L}U\right)}{h_0^2}\frac{\partial^2\tilde v}{\partial\tilde y^2}\right)$$

$$\frac{\rho h_0 U^2}{L^2}\frac{D\tilde v}{D\tilde t} = -\frac{\mu U L}{h_0^3}\frac{\partial\tilde p}{\partial\tilde y} + \frac{\mu U}{h_0 L}\left(\left(\frac{h_0}{L}\right)^2\frac{\partial^2\tilde v}{\partial\tilde x^2} + \frac{\partial^2\tilde v}{\partial\tilde y^2}\right)$$

$$\underbrace{\frac{\rho U h_0}{\mu}}_{Re}\left(\frac{h_0}{L}\right)\frac{D\tilde v}{D\tilde t} = -\left(\frac{L}{h_0}\right)^2\frac{\partial\tilde p}{\partial\tilde y} + \left(\left(\frac{h_0}{L}\right)^2\frac{\partial^2\tilde v}{\partial\tilde x^2} + \frac{\partial^2\tilde v}{\partial\tilde y^2}\right)$$

$\frac{h_0}{L} = \varepsilon \ll 1$; Also assume (as before) that $\varepsilon Re \ll 1$.

$$Re\,\varepsilon\frac{D\tilde v}{D\tilde t} = \underbrace{-\left(\frac{1}{\varepsilon}\right)^2\frac{\partial\tilde p}{\partial\tilde y}} + \varepsilon^2\frac{\partial^2\tilde v}{\partial\tilde x^2} + \frac{\partial^2\tilde v}{\partial\tilde y^2}$$

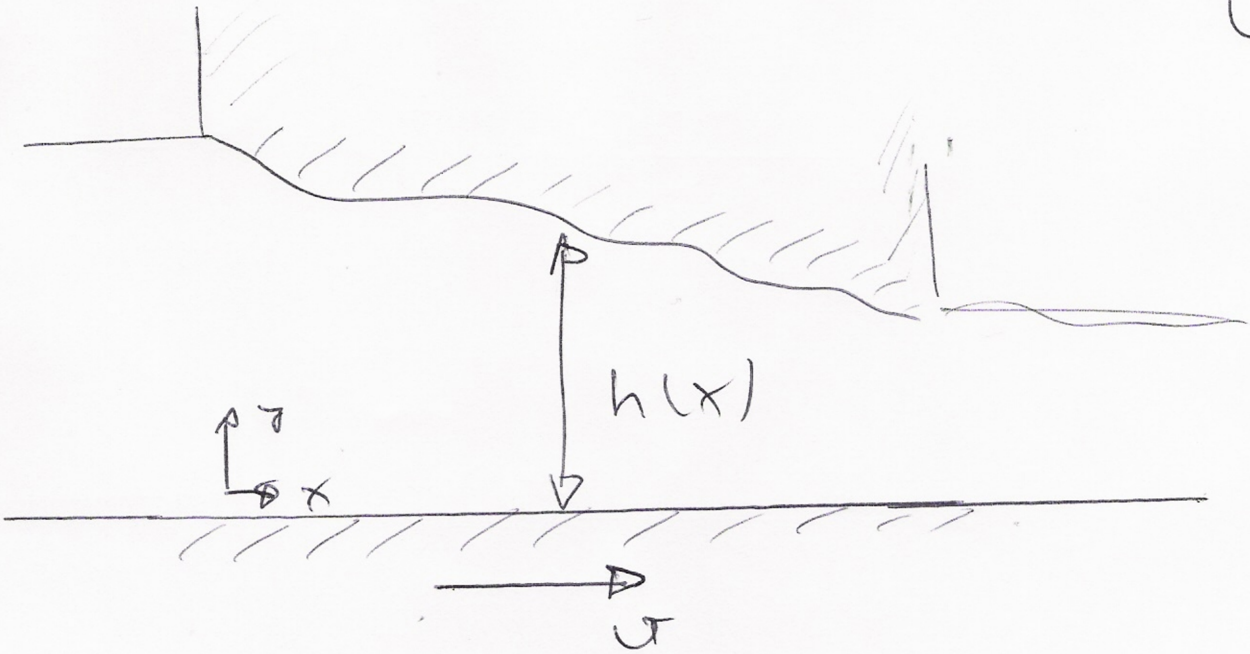
This term is $O\left(\frac{1}{\varepsilon^2}\right) \gg 1$
larger than all other terms.

There are no undetermined scales left, so to leading order (i.e, neglecting terms of size $Re\,\varepsilon$ and ε^2 against terms of size "1") the Navier-Stokes eqns can be approximated by

$$\boxed{\begin{gathered}\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \\ \& \\ \frac{\partial p}{\partial y} = 0\end{gathered}}$$

In other words, the wall slope changes so slowly that locally the flow behaves as if it were a parallel flow through a channel of constant width.

2)



$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

$u(y=0) = U$

$u(y=h(x)) = 0$

Since $\frac{\partial p}{\partial y} = 0$ we can integrate w.r.t. y:
& apply the BC:

$$u(x,y) = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\, y\,(h(x)-y) + U\left(1-\frac{y}{h(x)}\right)$$

Additional requirement: The same volume flux has to pass through every cross section.

$Q = \int_0^{h(x)} u(x,y)\,dy = \text{const.}$ (and - as yet - unknown)

$$Q = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) + U\left(h - \frac{h}{2}\right)$$

$$Q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} + \frac{1}{2}Uh$$

So:

$$\frac{\partial p}{\partial x} = \frac{12\mu}{h^3}\left(\frac{h}{2}U - Q\right)$$

integrate w.r.t. x & use $p(x=0) = p_0$:

$$p(x) = p_0 + 6\mu \int_0^x \left(\frac{U}{h^2(s)} - \frac{2Q}{h^3(s)}\right) ds$$

The second pressure B.C. provides an eqn. for Q:

$p(L) = p_0$:

$$\int_0^L \left(\frac{U}{h^2(s)} - \frac{2Q}{h^3(s)}\right) ds = 0$$

$$Q = \frac{1}{2}U \frac{\int_0^L h^{-2}(s)\,ds}{\int_0^L h^{-3}(s)\,ds}$$

(ii) Now for $h(x) = h_L + \frac{x}{L}(h_R - h_L)$

$$\int_0^L \left(h_L + \frac{x}{L}(h_R - h_L)\right)^{-N} dx =$$

$$= \frac{L}{h_R - h_L} \int_{h_L}^{h_R} z^{-N}\, dz = \frac{L}{h_R - h_L} \frac{1}{(-N+1)} \left(h_R^{-N+1} - h_L^{-N+1}\right)$$

So:

$$Q = \frac{1}{2} \upsilon \frac{\left(-\frac{1}{1}\right)\left(\frac{1}{h_R} - \frac{1}{h_L}\right)}{\left(-\frac{1}{2}\right)\left(\frac{1}{h_R^2} - \frac{1}{h_L^2}\right)} = \upsilon \frac{h_R^2 h_L^2 (h_L - h_R)}{(h_L^2 - h_R^2)\, h_R h_L}$$

$$\underline{\underline{Q = \upsilon \frac{h_R h_L}{h_L + h_R}}}$$

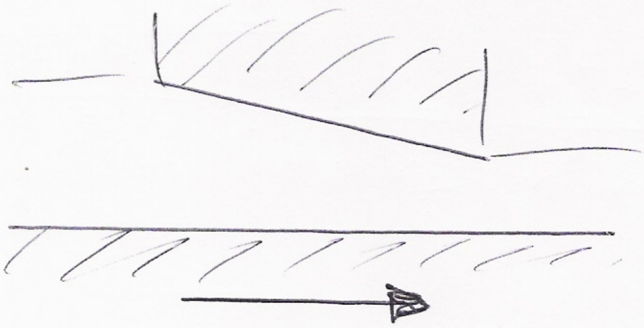
$$= \frac{1}{(h_R^2 - h_L^2)\, h^2 h_L} \left[\cancel{h^2 h_R} + h^2 h_L - h h_L h_R - h h_L^2 + h_R h_L^2 - \cancel{h_R h^2} \right]$$

$$= \frac{1}{(h_R^2 - h_L^2)\, h^2} \underbrace{\left(h^2 - h h_R - h h_L + h_R h_L \right)}_{(h_L - h)(h_R - h)}$$

$$\frac{p(x) - p_0}{6 \mu U L} = \frac{(h_R - h(x))\,(h_L - h(x))}{(h_R^2 - h_L^2)\, h^2(x)}$$

Now: $h(x)$ lies between h_R & h_L so the numerator of this expression is neg.

Hence if $h_R < h_L$, i.e:



$p(x) > 0$ under the bearing.

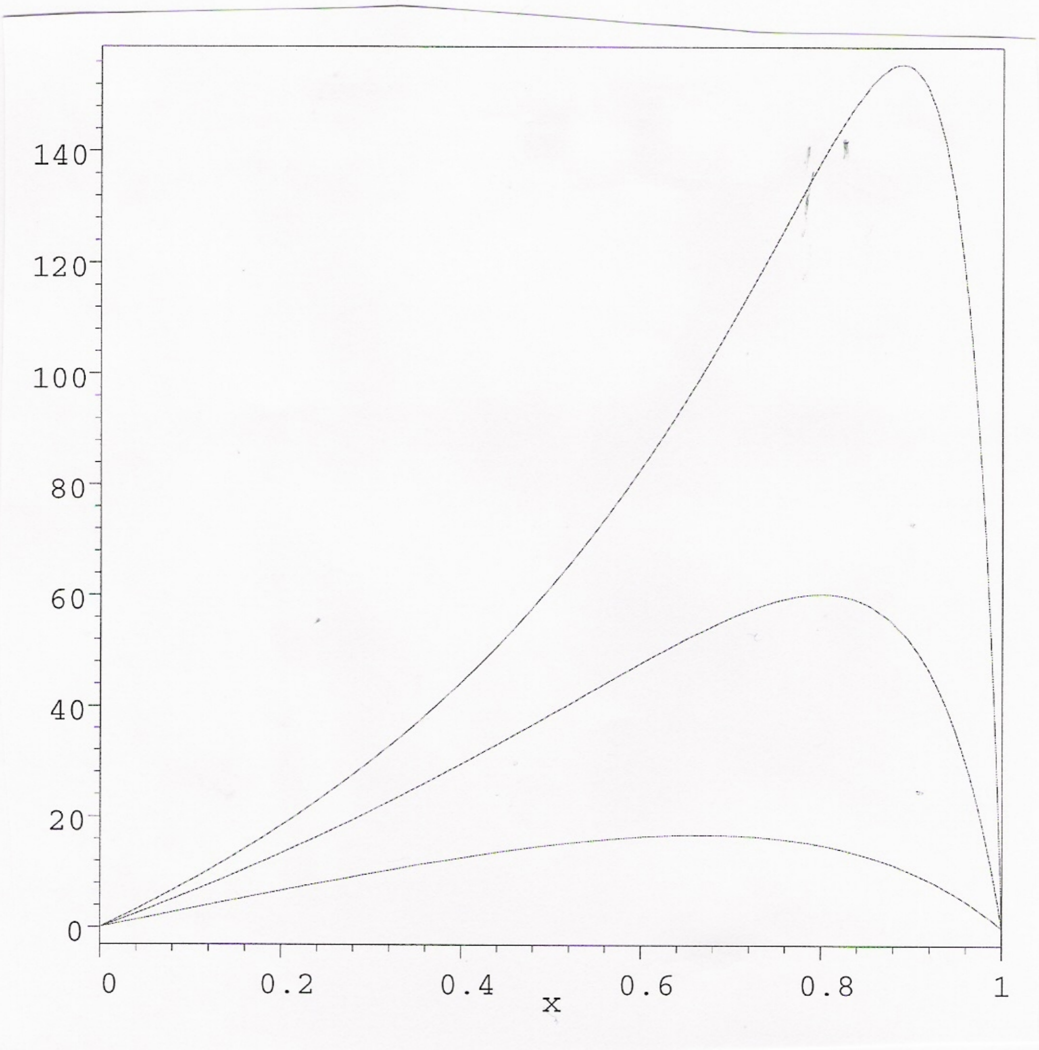
As the fluid is drawn into the narrowing lubrication gap the pressure exceeds the external pressure & produces a net upwards force, which keeps the surfaces from coming into contact.

Note that if the average height of the lubrication gap is h_0 then:

$$\frac{p(x)-p_0}{6\mu UL} \sim \frac{1}{h_0^2}$$

Hence, tremendous pressures can be generated in narrow gaps.

$$\frac{p(x) - p_0}{6\mu U L}$$



for $L = 1$, $h_L = 0.1$ &

$h_R = 0.05$, 0.025 & 0.0125