EXAMPLE SHEET III

1) Incompressibility requires

$$\operatorname{div} \underline{u} = \frac{\partial u_i}{\partial x_i} = 0$$

$$0 \overset{!}{=} -x_2^2 \sin x_1 + 3A x_2^2 \sin x_1$$

$$\underline{\underline{A = \frac{1}{3}}}$$

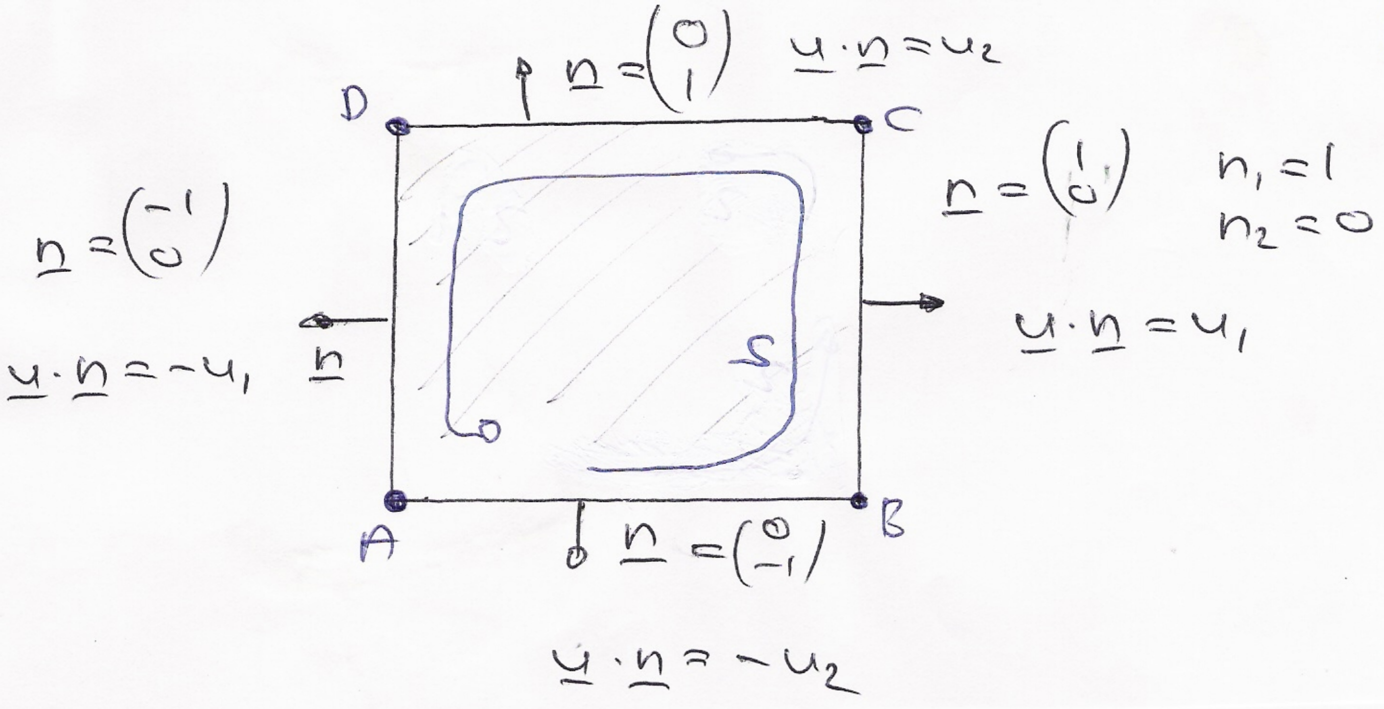
2) integral form of the equation of continuity:

$$\boxed{\oint \rho \, \underline{u} \cdot \underline{n} \, ds = \int \frac{d\rho}{dt} \, dA} \qquad (1)$$

Note that we don't know what ρ is! So let's assume that $\rho = \text{const.}$ i.e. the fluid is incompressible & check if the flow field is consistent with this assumption.

For $\rho = \text{const.}$:

$$\oint \rho \, \underline{u} \cdot \underline{n} \, ds = \rho \underbrace{\int \underline{u} \cdot \underline{n} \, ds}_{\text{this can be evaluated}}$$



$$\oint \underline{u}\cdot\underline{n}\,ds = \int_A^B \underline{u}\cdot\underline{n}\,ds + \int_B^C \underline{u}\cdot\underline{n}\,ds + \int_C^D \underline{u}\cdot\underline{n}\,ds + \int_D^A \underline{u}\cdot\underline{n}\,ds$$

on AB: $ds = dx_1$ $\quad \int_A^B ds = \int_{x_1=0}^{1} dx_1$

BC: $ds = dx_2$ $\quad \int_B^C ds = \int_{x_2=0}^{1} dx_2$

CD: $ds = -dx_1$ $\quad \int_C^D ds = \int_{x_2=1}^{0} (-dx_1) = \int_{x_1=0}^{1} dx_1$

DA: $ds = -dx_2$ $\quad \int_D^A ds = \int_{x_2=1}^{0} (-dx_2) = \int_{x_2=0}^{1} dx_2$

$\oint \underline{u} \cdot \underline{n} \, ds =$

$\int_0^1 -u_2(x_2=0)\, dx_1 + \int_0^1 u_1(x_1=1)\, dx_2 +$

$\int u_2(x_2=1)\, dx_1 + \int -u_1(x_1=0)\, dx_2$

$= \int_0^1 (4+3x_1^2)\, dx_1 + \int_0^1 (3+x_2+x_2^3)\, dx_2 +$

$+ \int_0^1 (4-\frac{1}{2}x_1+3x_1^2)\, dx_1 + \int_0^1 -(3+x_2)\, dx_2$

$= -(4+3\frac{1}{3}) + (3+\frac{1}{2}+\frac{1}{4})$

$+(4-\frac{1}{2}\frac{1}{2}+3\,\frac{1}{3}) - (3+\frac{1}{2})$

$= -\frac{20}{4} + \frac{15}{4} + \frac{19}{4} - \frac{14}{4} = 0$

$\oint \underline{u} \cdot \underline{n} \, ds = 0$

This is <u>consistent</u> with the assumption of $\rho =$ const. [it's a necessary condition for incompressibility but not a sufficient one]. It does not rule out the possibility of local variations in fluid density! Conversely if we had obtained $\int \underline{u}\cdot\underline{n}\, ds \neq 0$ then we could have concluded that the fluid <u>must</u> be compressible.

Another way of seeing this is via Gauss' theorem (which holds for <u>any</u> vector field $\underline{u}$)

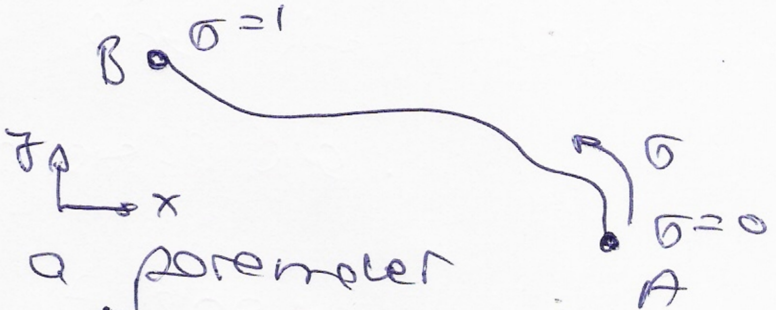
$$\int \underline{u}\cdot\underline{n}\, ds = \int \nabla\cdot\underline{u}\, dA$$

If this is zero then the integral of $\nabla\cdot\underline{u}$ has to vanish but this does <u>not</u> imply that $\nabla\cdot\underline{u} \equiv 0$ at every point in the domain!

Appendix to Q2:

Formal evaluation of line integrals: Assume the boundary of the domain is given in the form

$$\begin{pmatrix} x(\sigma) \\ y(\sigma) \end{pmatrix}$$

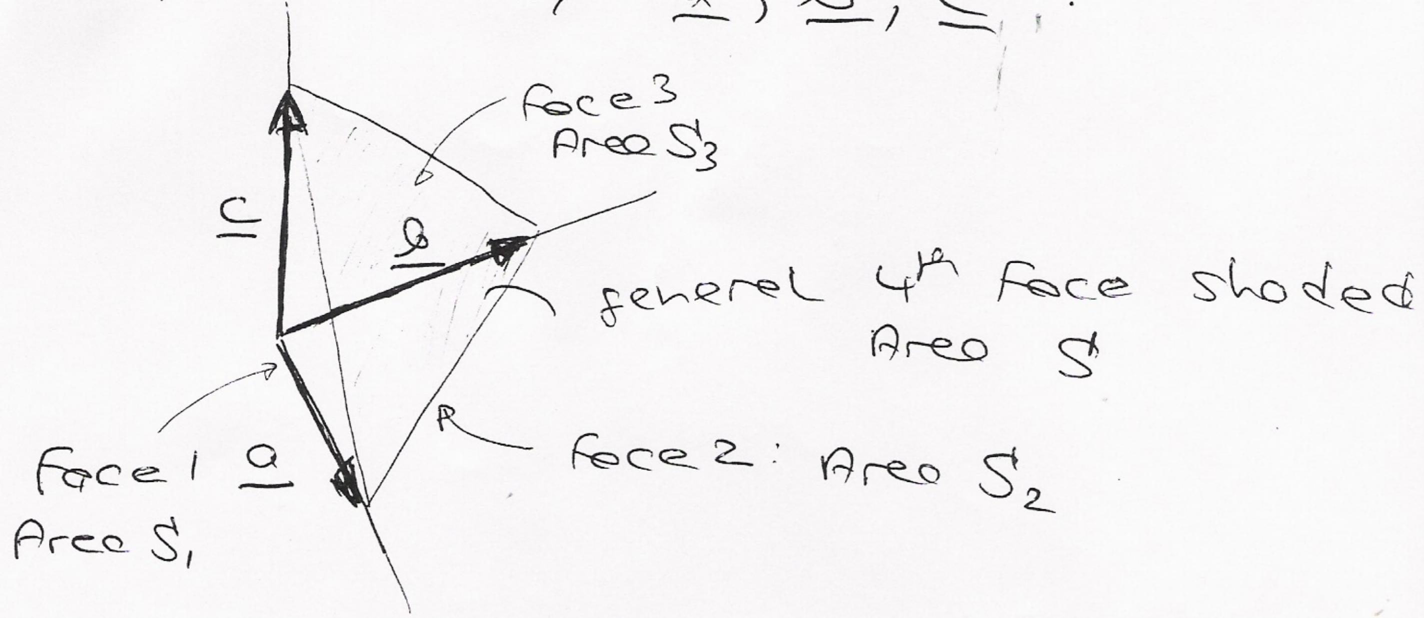


where σ is a parameter (not necessarily the arclength!)

Then:

$$\int_A^B f(x,y)\,ds = \int_{\sigma=0}^{\sigma=1} f(x(\sigma),y(\sigma))\underbrace{\sqrt{\left(\frac{dx}{d\sigma}\right)^2+\left(\frac{dy}{d\sigma}\right)^2}\,d\sigma}_{ds}$$

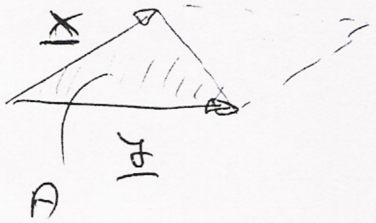
3) Consider finite tetrahedron spanned by $\underline{a}, \underline{b}, \underline{c}$:



Any triangular face spanned by $\underline{x}, \underline{y}$:

$$|\underline{x} \times \underline{y}| = 2A$$

area of triangle



& $\underline{x} \times \underline{y} \perp \underline{x}, \underline{y}$

2 Cross product of two vectors spanning the general face:

$$S\,\underline{n} = \tfrac{1}{2}(\underline{a} - c) \times (\underline{b} - \underline{c})$$

$$= \tfrac{1}{2}(\underline{a} \times \underline{b} - \underline{c} \times \underline{b} - \underline{a} \times \underline{c} - \cancel{\underline{c} \times \underline{c}})$$

$$S\,\underline{n} + \underbrace{\tfrac{1}{2}\underline{b} \times \underline{a}} + \underbrace{\tfrac{1}{2}\underline{c} \times \underline{b}} + \underbrace{\tfrac{1}{2}\underline{a} \times \underline{c}} = 0$$

$$S\,\underline{n} + S_2\underline{n}_2 + S_3\underline{n}_3 + S_1\underline{n}_1 = 0$$

q.e.d.