EXAMPLE SHEET V

1) Following the derivation in the lecture:

$$\boxed{0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha} \quad (1)$$

$$\boxed{0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha} \quad (2)$$

$$\boxed{0 = -\frac{\partial p}{\partial z}} \rightarrow p = p(x,y) \quad (3)$$

BC: $y=0$: No slip on belt:

$$\boxed{u(y=0) = -U} \quad (4)$$

$y=h$: free surface, pressure equal to zero & tangential shear in neg. x-direction:

Applied traction onto fluid is

$\underline{t} = -\tau_0 \underline{e_x}$, i.e.

$t_1 = -\tau_0$; $t_2 = 0$

Traction BC:

$$t_i = \tau_{ij} n_j$$

$\hookrightarrow$ outer unit normal on fluid:

$\underline{n} = \underline{e_y}$:

$n_1 = 0 \quad n_2 = 1$

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$t_i = -pn_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

$\underline{i=1}$:

$$t_1 = -\tau_0 = \underbrace{\mu\left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1}\right) n_j}$$

only $j=2$ gives contribution to sum over j

$$-\tau_0 = \mu \frac{\partial u_1}{\partial x_2} \qquad \text{since } u_2 = 0$$

$$\boxed{-\tau_0 = \mu \frac{\partial u}{\partial y}} \qquad (5)$$

@ $y = h$

$i=2$:

$$t_2 = 0 = -p + \underbrace{\mu\left(\frac{\partial u_2}{\partial x_j} + \frac{\partial u_j}{\partial x_2}\right) n_j}$$

no contribution since either $n_j = 0$ (for $j=1$) or $u_2 = 0$ (for $j=2$) cancels entire expression

$$\boxed{0 = -p} \quad (6)$$

@ $y = h$

Now integrate (2) w.r.t y & use BC. (6) at free surface ($y=h$)

$$p(x,y) = -\rho g \cos\alpha \, y + f(x)$$

→ const. of integration can depend on x.

$p(y=h) = 0 \rightarrow f(x) = \rho g \cos\alpha \, h$

$$p(x,y) = \rho g \cos\alpha \,(h - y)$$ (hydrostatic press. distrib.)

$p(x,y) = p(y)$ into (1) : $\frac{\partial p}{\partial x} = 0$

$$0 = \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha$$

$$u(y) = -\frac{1}{2}\underbrace{\frac{\rho}{\mu}}_{\frac{1}{\nu}} g \sin\alpha \, y^2 + Ay + B$$

B.C.

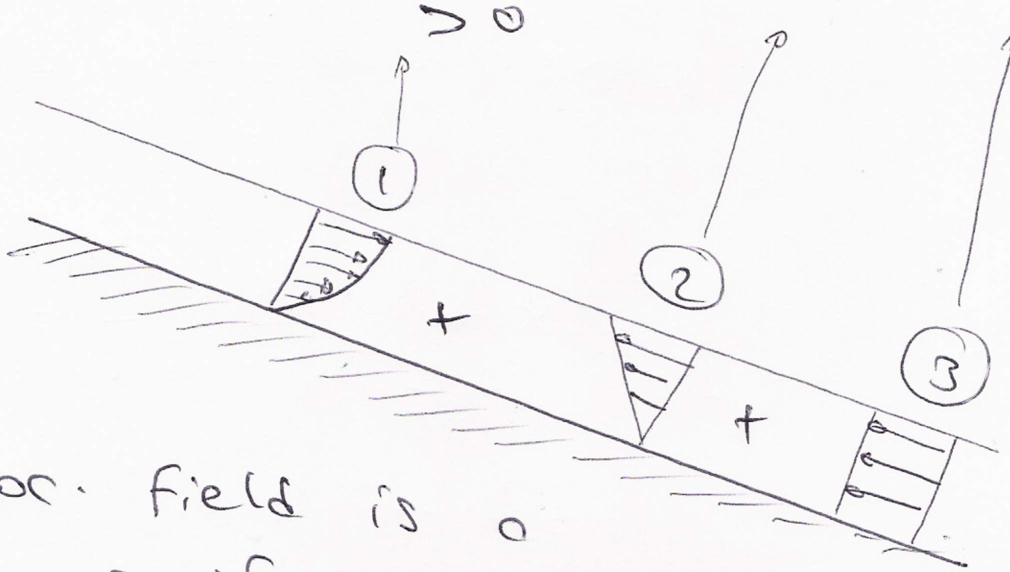
$$u(0) = -U = B$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=h} = -\frac{\tau_0}{\mu}$$

$$-\frac{\tau_0}{\mu} = -\frac{1}{\nu} g \sin\alpha \, h + A$$

$$A = \frac{1}{\nu} g \sin\alpha \, h - \frac{\tau_0}{\mu}$$

$$u(y) = \frac{g \sin\alpha}{\nu}\underbrace{\left[hy - \frac{1}{2}y^2\right]}_{>0} - \frac{\tau_0}{\mu}y - U$$



veloc. field is a superpos. of

① : Gravity driven flow

② : Shear driven flow

③ : uniform rigid body flow due to belt motion ("plug flow")

(ii) $Q = \int_0^h u\,dy$

$$Q = \frac{g \sin\alpha}{\nu}\left(h\frac{h^2}{2} - \frac{1}{2}\frac{h^3}{3}\right) - \frac{\tau_0 h^2}{2\mu} - Uh$$

$$Q = \frac{1}{3}\frac{g h^3 \sin\alpha}{\nu} - \frac{\tau_0 h^2}{2\mu} - Uh$$

(iii) for $\sigma = 0$ & $\tau_0 = 0$

$Q > 0$. increasing τ_0 reduces Q until downword flow (driven by gravity) is compensated for by upword flow (driven by shear τ_0).

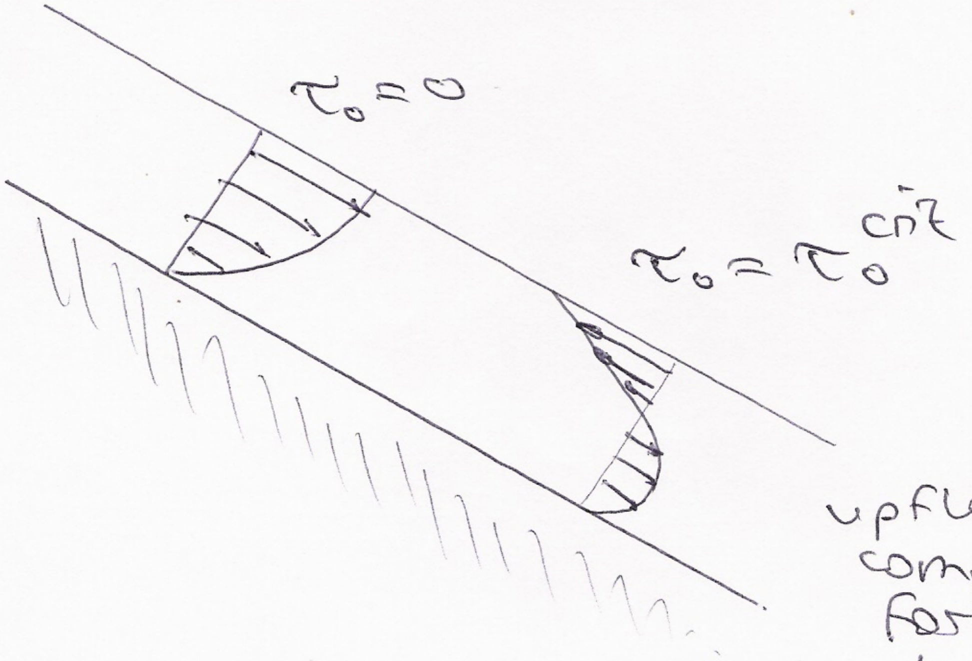
critical case:

$Q = 0$:

$$\tau_0^{crit} = \frac{2}{3} \frac{gh \sin\alpha \, \mu}{\nu}$$

$$\nu = \frac{\mu}{\rho}$$

$$\tau_0^{crit} = \frac{2}{3} \rho g h \sin\alpha$$



upflow compensates for downflow.

2) (i) The assumed velocity distribution is the simplest extension of the v-velocity on the two walls to the interior. The BC. do not depend on time or on x & z so we try a solution which only depends on y. u will have to vary with y because of the no slip cond. on the wall & the nonzero veloc. in the interior.

(ii) x-NSt:

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + v\frac{\partial u}{\partial y} = -\frac{G}{\rho} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

$u=u(y)$ $\quad v=V$ $\quad u=u(y)$

$$\boxed{\frac{G}{\rho} = \nu\frac{\partial^2 u}{\partial y^2} + V\frac{\partial u}{\partial y}}$$

y-NSE:

$$\cancel{\frac{\partial v}{\partial t}} + u\cancel{\frac{\partial v}{\partial x}} + v\cancel{\frac{\partial v}{\partial y}} = -\frac{1}{\rho}\cancel{\frac{\partial p}{\partial y}} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right)$$

$v = -V = \text{const.}$ $\quad$ 0 $\quad$ $v = -V = \text{const.}$

$\nabla p \cdot \underline{e}_y = 0$

$$\boxed{0 = 0} \quad \checkmark$$

Continuity

$$\cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial v}{\partial y}} = 0$$

$u = u(y)$ $\quad$ $v = -V = \text{const.}$

$$\boxed{0 = 0} \quad \checkmark$$

(iii) $$\frac{G}{\rho} = \nu\frac{\partial^2 u}{\partial y^2} + V\frac{\partial u}{\partial y} \qquad u(0) = u(h) = 0$$

2nd order ODE const. coeffns.

Hom. soln: $u_H \sim e^{\lambda y}$

$$0 = \nu\lambda^2 + V\lambda = \lambda(\nu\lambda + V) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = -\frac{V}{\nu}$$

$$u_H = A + B e^{-\frac{Vy}{\nu}}$$

Particular soln:

RHS = const.

Hom. soln. already contains a constant term so the particular soln must be of the form const $\times y$

Try: $u_p = C'y$

into ODE:

$$\frac{G}{\rho} = VC' \quad ; \quad C' = \frac{G}{\rho V}$$

$$u(y) = \frac{G}{\rho V} y + A + B e^{-\frac{Vy}{\nu}}$$

BC:

$u(0) = 0 = A + B \quad ; \quad A = -B$

$$u(h) = 0 = \frac{Gh}{\rho V} + A + B e^{-\frac{Vh}{\nu}}$$

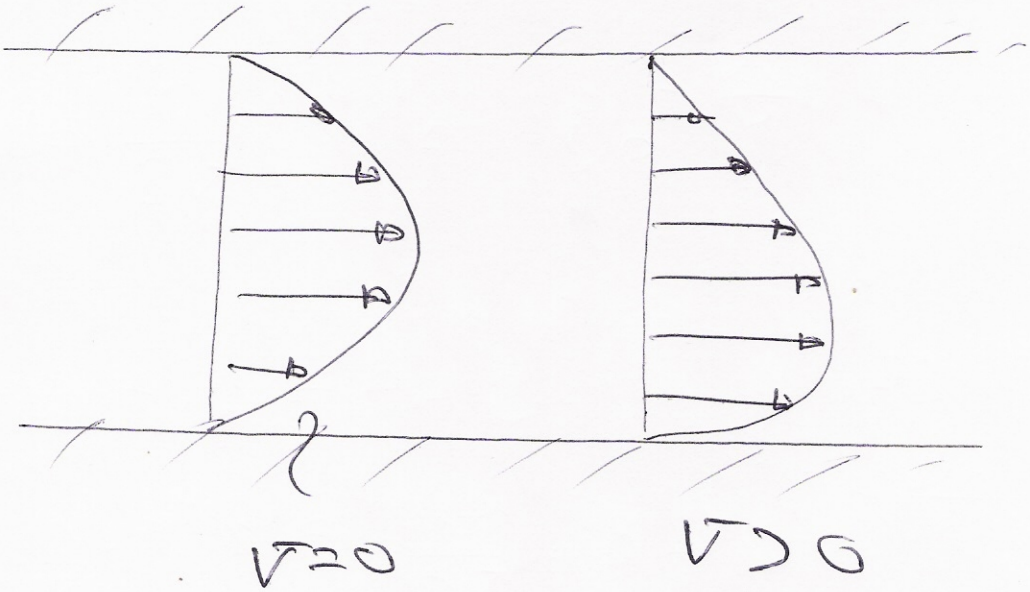
$$\frac{Gh}{\rho V} = B\left(1 - e^{-\frac{Vh}{\nu}}\right)$$

$$B = -A = \frac{Gh}{\rho V}\left(1 - e^{-\frac{Vh}{\nu}}\right)^{-1}$$

$$u(y) = \frac{Gy}{\rho V} + \frac{Gh}{\rho V}\,\frac{e^{-\frac{Vy}{\nu}} - 1}{1 - e^{-\frac{Vh}{\nu}}}$$

$$\underline{\underline{u(y) = \frac{G}{\rho V}\left(y - h\,\frac{1 - e^{-\frac{Vy}{\nu}}}{1 - e^{-\frac{Vh}{\nu}}}\right)}}$$

The presence of the transverse flow breaks the symmetry of the flow:



see animation at:

http://www.maths.man.ac.uk/~mheil/Lectures/Fluids/MT4261-Fluids.html

Additional comments:

For $V \to 0$ $u(y)$ should approach the usual parabolic Pousseuille profile.

Expand

$$e^{-V\alpha} = 1 - V\alpha + \frac{1}{2}V^2\alpha^2 - \dots$$

$$u(y) = \frac{G}{\beta V}\left(y - h\,\frac{1-\left(1 - V\frac{y}{\nu} + \frac{1}{2}V^2\left(\frac{y}{\nu}\right)^2 - \dots\right)}{1-\left(1 - V\frac{h}{\nu} + \frac{1}{2}V^2\left(\frac{h}{\nu}\right)^2 - \dots\right)}\right)$$

$$= \frac{G}{\beta V}\left(y - h\,\frac{\cancel{\frac{V}{\nu}}\left(y - \frac{1}{2}V\frac{y^2}{\nu} + \dots\right)}{\cancel{\frac{V}{\nu}}\left(h - \frac{1}{2}V\frac{h^2}{\nu} + \dots\right)}\right)$$

$$= \frac{G}{\beta \cancel{V}}\left(\frac{\cancel{yh} - \frac{1}{2}\cancel{V}\frac{yh^2}{\nu} + \dots \cancel{- hy} + \frac{1}{2}\cancel{V}\frac{y^2 h}{\nu} - \dots}{\left(h - \frac{1}{2}V\frac{h^2}{\nu}\right)}\right)$$

Now $V \to 0$

$$u(y) = \frac{G}{\beta\nu}\left(-\frac{1}{2}yh + \frac{1}{2}y^2\right) = -\frac{1}{2}\frac{G}{\mu}(hy - y^2)$$

$\beta\nu \to \mu$

✓