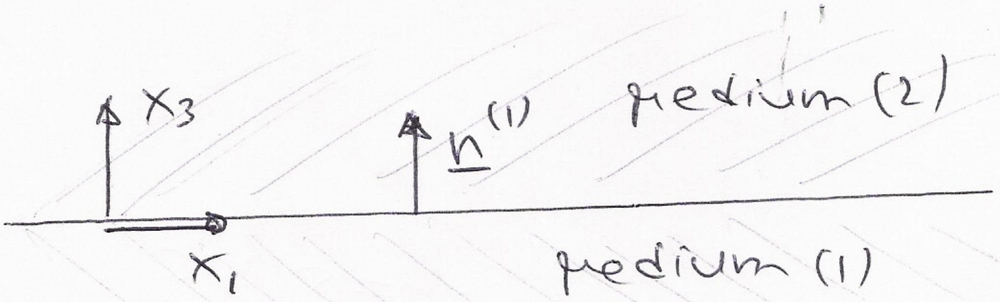
EXAMPLE SHEET IV



1) (i) $\underline{n}^{(1)} = \underline{e}_3 = \underline{k}$

$n_1^{(1)} = 0 \quad n_2^{(1)} = 0 \quad n_3^{(2)} = 1$

Traction exerted onto a continuum is given by

$$t_i = \tau_{ij} n_j$$

here: $\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

(Newtonian fluid)

$$t_i = -p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

$\underline{i=1}$: $t_1 = -p \overset{0}{\cancel{n_1}} + \mu \underbrace{\left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1}\right) n_j}$

$n_1 = 0$

in the summation only $j = 3$ gives a contribution from $n_3 = 1$

$$t_1 = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

Similarly

$i=2$:
$$t_2 = \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$i=3$:
$$t_3 = -p n_3 + \mu\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\right) n_3$$

$$t_3 = -p + 2\mu \frac{\partial u_3}{\partial x_3}$$

(ii) dynamic / traction BC.

$t_i^{(1)} = \tau_{ij}^{(1)} n_j = \tau_{ij}^{(2)} n_j$ using the same unit normal on both sides.

Medium (2) inviscid:

$$\tau_{ij}^{(2)} = -p^{(2)} \delta_{ij} = -P\, \delta_{ij}$$

$$\Rightarrow \tau_{ij}^{(2)} n_j = -P\, \delta_{ij} n_j = -P n_i$$

(only traction normal to the interface)

∂ For $i=3$:

$$t_3 = -P$$

$$-p + 2\mu \frac{\partial u_3}{\partial x_3} = -P$$

$$\underline{\underline{p = P + 2\mu \frac{\partial u_3}{\partial x_3}}}$$

The fluid pressure in the viscous fluid exceeds the pressure in the inviscid fluid by the "viscous normal stress" $2\mu \frac{\partial u_3}{\partial x_3}$.

[Physically this term represents the viscous diffusion of momentum in the direction of the flow; it is similar in nature to the shear stresses $\mu \frac{\partial u_1}{\partial x_3}$ (say) which represent the viscous diffusion of momentum in the direction transverse to the flow. See e.g. Landau & Lifshitz "Fluid Mechanics", Pergamon Press; 2nd ed. p. 44]

The viscous normal stresses are important in free surface flows.

(iii) The traction acting on the surface was determined in (i).

The no-slip & no-penetration conditions imply

$u_i \equiv 0$ for $x_3 = 0$ i.e. for <u>all</u> x_1, x_2.

therefore:

$$\frac{\partial u_i}{\partial x_1} = \frac{\partial u_i}{\partial x_2} = 0$$

now using the continuity eqn.

$$\frac{\partial u_i}{\partial x_i} = \underbrace{\cancel{\frac{\partial u_1}{\partial x_1}}}_{0} + \underbrace{\cancel{\frac{\partial u_2}{\partial x_2}}}_{0} + \frac{\partial u_3}{\partial x_3} = 0$$

shows that on the surface

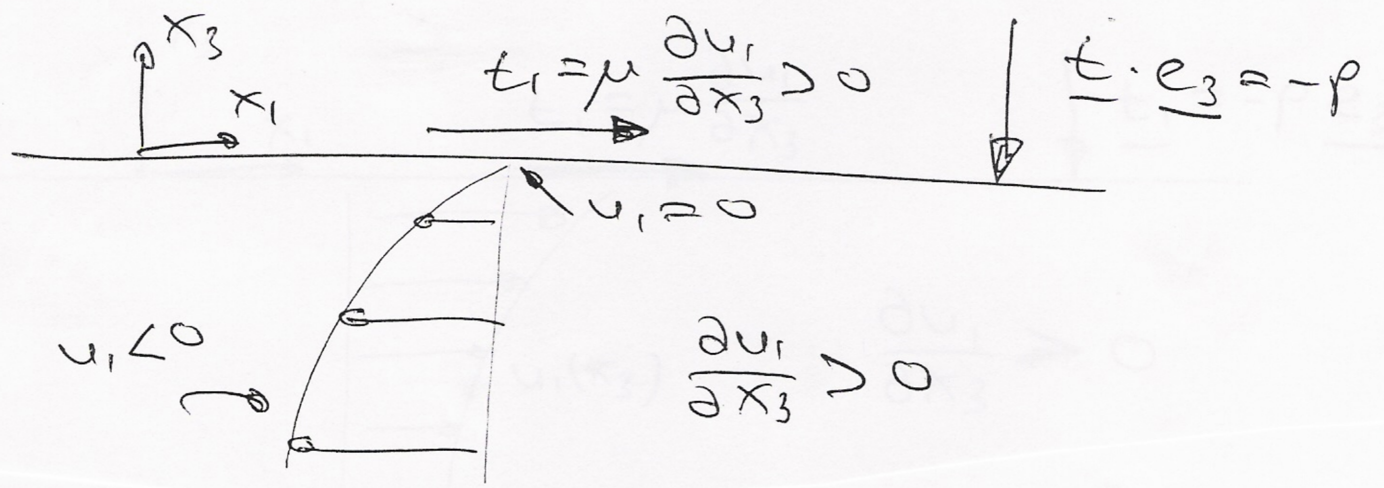
$\frac{\partial u_3}{\partial x_3} = 0$ as well.

Hence the traction exerted onto the fluid is:

$$t_1 = \mu \frac{\partial u_1}{\partial x_3} \qquad t_2 = \mu \frac{\partial u_2}{\partial x_3}$$

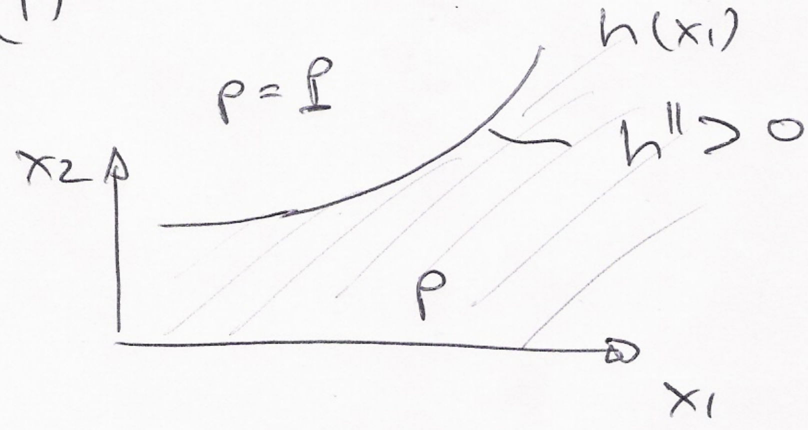
and $t_3 = -p$.

Illustration:



- In this sketch the shear stress applied to the fluid (towards the right as $t_1 > 0$) drags the fluid in that direction: The u_1-velocity at the interface is larger than in interior, i.e. $\partial u_1 / \partial x_3 > 0$
- The pressure "presses against" the interface
- Note: At no-slip boundaries there are no viscous normal stresses.

2) (i)



Hydrostatics: $u_i = 0$

$$\tau_{ij} = -p\delta_{ij}$$

inviscid fluid

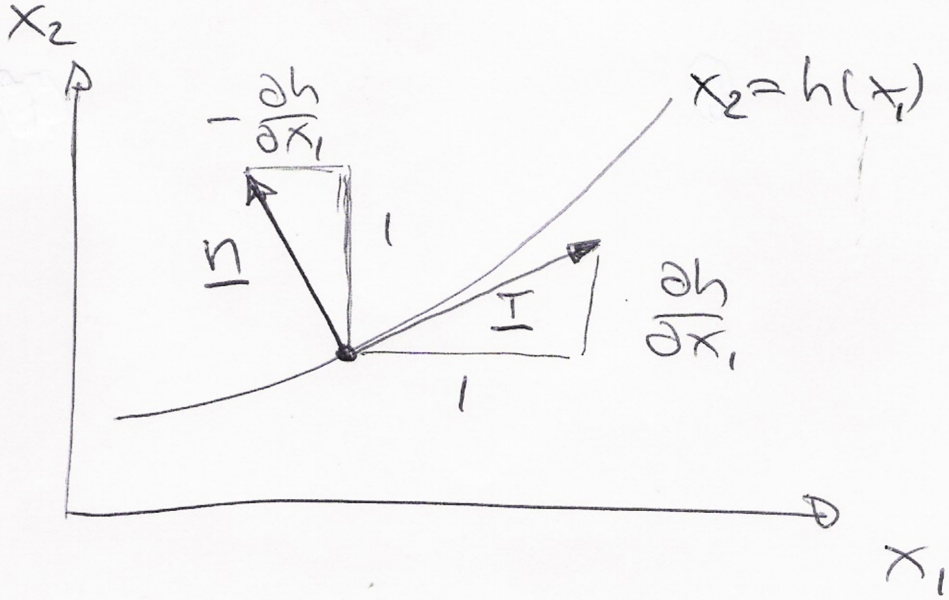
$$-\mathcal{P} n_i = -p\delta_{ij} n_j + \sigma\kappa n_i$$

$$\underline{\underline{p = \mathcal{P} + \sigma\kappa}}$$

For the interface shown above, the centre of curvature is inside the inviscid fluid hence its pressure must be larger than the pressure in the viscous fluid and the sign of the curvature has to be chosen as

$$\kappa = -\frac{h''}{\left(1+(h')^2\right)^{3/2}}$$

(ii)



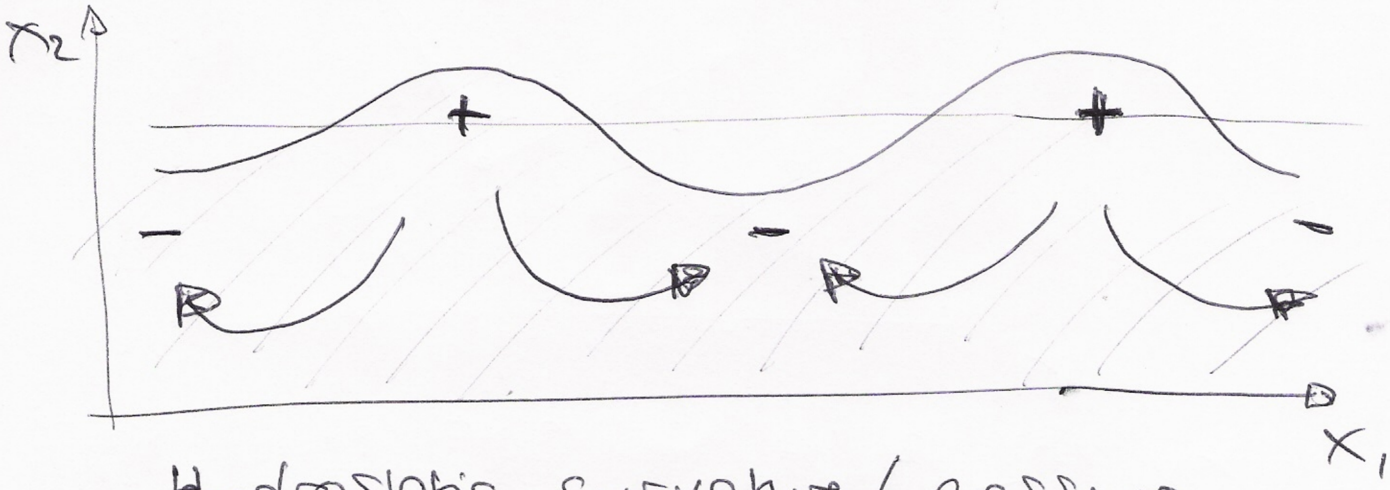
From the sketch: a tangent vector $\underline{T}$ to the interface is given by $\left(1, \frac{\partial h}{\partial x_1}\right)$. Hence the outer unit normal $\underline{n}$ is

$$\underline{n} = \frac{1}{\sqrt{1+\left(\frac{\partial h}{\partial x_1}\right)^2}} \begin{pmatrix} -\frac{\partial h}{\partial x_1} \\ 1 \end{pmatrix}$$

or
$$\underline{\underline{n_1 = -\left(1+\frac{\partial h}{\partial x_1}\right)^{-1/2} \frac{\partial h}{\partial x_1}}}$$

$$\underline{\underline{n_2 = \left(1+\frac{\partial h}{\partial x_1}\right)^{-1/2}}}$$

(iii) Surface-tension-driven flows



Hydrostatic curvature / pressure distribution indicated by +/− for high / low fluid pressure.

This drives a flow from regions of high pressure to regions of low pressure (as indicated) and tends to level the surface.

for $h', h'', \kappa \ll 1$

$$n_1 \approx -\frac{\partial h}{\partial x_1} = O(\varepsilon)$$

$$n_2 \approx 1 = O(1) \gg n_1$$

$$\kappa \approx -\frac{\partial^2 h}{\partial x_1^2} = O(\varepsilon)$$

Dynamic B.C. with $\underline{f} = 0$:

$$0 = \tau_{ij} n_j + \sigma \kappa n_i$$

$$0 = -p n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j + \sigma \kappa n_i$$

<u>$i=1$:</u>

$$0 = \underbrace{+p\overset{n_1}{\frac{\partial h}{\partial x_1}}}_{\substack{\varepsilon\ \varepsilon\\ \varepsilon^2}} \underbrace{- 2\mu \frac{\partial u_1}{\partial x_1}\overset{n_1}{\frac{\partial h}{\partial x_1}}}_{\substack{\varepsilon\ \ \varepsilon\\ \varepsilon^2}} \underbrace{+ \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\overset{n_2}{1}}_{\substack{\varepsilon\\ \varepsilon}} \underbrace{- \sigma\kappa\overset{n_1}{\frac{\partial h}{\partial x_1}}}_{\substack{\varepsilon\,\varepsilon\\ \varepsilon^2}}$$

Size of terms

∂ Dominant term for $\varepsilon \ll 1$:

$$\underline{\underline{\mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0}}$$ [no shear].

$\underline{i=2}$

$$0 = \underbrace{-\underset{\varepsilon}{p}\cdot \overset{n_2}{1}}_{\varepsilon} - \underbrace{\mu \underset{\varepsilon}{\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)}\overset{n_1}{\underset{\varepsilon}{\frac{\partial h}{\partial x_1}}}}_{\varepsilon^2} + \underbrace{2\mu\,\underset{\varepsilon}{\frac{\partial u_2}{\partial x_2}}\,\overset{n_2}{1}}_{\varepsilon} + \underbrace{\underset{\varepsilon}{\sigma\kappa}\,\overset{n_2}{1}}_{\varepsilon}$$

So dominant term for $\varepsilon \ll 1$:

$$(p - \sigma\kappa) = 2\mu \frac{\partial u_2}{\partial x_2}$$

or

$$\underline{\underline{p + \sigma \frac{\partial^2 h}{\partial x_1^2} = 2\mu \frac{\partial u_2}{\partial x_2}}}$$

Pressure jump over the interface is caused by the viscous normal stress (see question 1) & the surface tension.