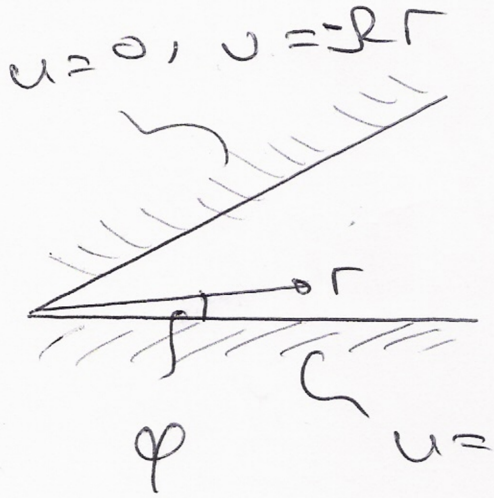
EXAMPLE SHEET VIII

1) (i) $\nabla^4 \psi = 0$

& no slip on both walls.



$u = u_r$

$v = u_\varphi$

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \qquad v = -\frac{\partial \psi}{\partial r}$$

No radial velocity @ $\varphi = 0, \alpha$

$$\boxed{\frac{\partial \psi}{\partial \varphi} = 0 \quad \text{at } \varphi = 0} \qquad (1)$$

$$\boxed{\frac{\partial \psi}{\partial \varphi} = 0 \quad \text{at } \varphi = \alpha} \qquad (2)$$

$v = 0$ at $\varphi = 0$

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{at } \varphi = 0$$

so $\psi = \text{const}$ at $\varphi = 0$

we follow the usual convention & set $\psi = 0$, at stationary impermeable walls:

$$\boxed{\psi = 0 \quad \text{at} \quad \varphi = 0} \quad (3)$$

Now:

$$u = -\Omega r = -\frac{\partial \psi}{\partial r} \quad \text{at} \quad \varphi = \cancel{\alpha}$$

integrate w.r.t r:

$$\psi = \frac{1}{2}\Omega r^2 + C \quad \text{at} \quad \varphi = \alpha$$

& to achieve consistency with (3):
$C = 0$, so:

$$\boxed{\psi = \frac{1}{2}\Omega r^2 \quad \text{at} \quad \varphi = \alpha} \quad (4)$$

(ii) The problem described by
$\nabla^4 \psi = 0$ & the 4 BC'
depends on: $r, \varphi, \alpha, \Omega$ so

$$\psi = \psi(r, \varphi; \alpha, \Omega)$$

The PDE & the BC (1)-(3) are homogeneous. BC (4) is linear in Ω. Since the PDE is linear we must have:

$$\psi = \Omega \tilde{f}(r, \varphi; \alpha)$$

Dimensions:

$$\underbrace{u}_{\frac{m}{sec}} = \frac{\partial \psi}{\partial y}$$

$$[\psi] = \frac{m}{sec} m = \frac{m^2}{sec}$$

$$[\Omega] = sec^{-1}$$

$$[r] = m$$

$$[\varphi] = [\alpha] = 1 \quad (\text{dim. less})$$

So to get coherent dimensions we must have

$$\psi = \Omega r^2 \underbrace{\hat{f}(r, \varphi; \alpha)}_{\text{dim. less}}$$

$\hat{f}(r, \varphi; \alpha)$ must be a dim. less fct. of its arguments.

$\rightarrow$ $\hat{f}$ must be indep. of r.

so: $\boxed{\psi = \Omega r^2 f(\varphi; \alpha)}$

(iii) $\nabla^4 = \nabla^2 \nabla^2$

$$\nabla^2 \frac{\psi}{\Omega} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) r^2 f(\varphi)$$

$$= 2 \cdot 1 \cdot f + 2f + f''$$

$$= 4f + f''$$

$$\nabla^4 \frac{\psi}{\Omega} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\left(4f(\varphi) + f''(\varphi)\right)$$

$$\boxed{\frac{d^2}{d\varphi^2}\left(4 + \frac{d^2}{d\varphi^2}\right) f(\varphi) = 0}$$

(iv) $4f'' + f^{IV} = 0$

$$f \sim e^{\lambda\varphi}$$

$$4\lambda^2 + \lambda^4 = 0$$

$$\lambda^2(4+\lambda^2) = 0$$

$\lambda_{12} = 0$ (double root)

$\lambda_{34} = \pm 2i$

$$\boxed{f(\varphi) = A + B\varphi + C\cos 2\varphi + D\sin 2\varphi}$$

$$f'(\varphi) = B - 2C\sin 2\varphi + 2D\cos 2\varphi$$

B.C:

$\frac{\partial \psi}{\partial \varphi} = 0$ at $\varphi = 0$:

$f'(0) = 0$: $\boxed{B + 2D = 0}$ (a)

$\frac{\partial \psi}{\partial \varphi} = 0$ at $\varphi = \alpha$

$f'(\alpha) = 0$: $\boxed{0 = B - 2C\sin 2\alpha + 2D\cos 2\alpha}$ (b)

$\psi = 0$ at $\varphi = 0$

$f(0) = 0$: $\boxed{A + C = 0}$ (c)

$\psi = \frac{1}{2} \Omega r^2$ at $\varphi = \alpha$

$f(\alpha) = \frac{1}{2}$: $\boxed{\frac{1}{2} = A + \alpha B + C \cos 2\alpha + D \sin 2\alpha}$ (d)

(a) into (b):

$$B(1 - \cos 2\alpha) = 2C \sin 2\alpha$$

$$2B \sin^2 \alpha = 4C \sin\alpha \cos\alpha$$

$$B = 2C \frac{1}{\tan\alpha}$$

with (c)

$$\boxed{B = \frac{-2A}{\tan\alpha}}$$

(a):

$$D = -\frac{1}{2} B$$

$$\boxed{D = \frac{A}{\tan\alpha}}$$

(c): $\boxed{C = -A}$

into (d):

$$\frac{1}{2} = A\left(1 - \frac{2\alpha}{\tan\alpha} - \cos 2\alpha + \frac{\sin 2\alpha}{\tan\alpha}\right)$$

$$\underbrace{= A\left(\underbrace{1 - \cos 2\alpha}_{2\sin^2\alpha} - \frac{2\alpha}{\tan\alpha} + \underbrace{\frac{2\cancel{\sin\alpha}\cos\alpha\cos\alpha}{\cancel{\sin\alpha}}}\right)}_{2}$$

$$\frac{1}{2} = A\left(\frac{2\tan\alpha - 2\alpha}{\tan\alpha}\right)$$

$$A = \frac{\tan\alpha}{4(\tan\alpha - \alpha)}$$

$$B = -\frac{1}{2}\,\frac{1}{\tan\alpha - \alpha}$$

$$C = -\frac{\tan\alpha}{4(\tan\alpha - \alpha)}$$

$$D = \frac{1}{4(\tan\alpha - \alpha)}$$

$$\boxed{\psi = \frac{\Omega r^2}{4(\tan\alpha - \alpha)}\left(\tan\alpha(1 - \cos 2\varphi) + \sin 2\varphi - 2\varphi\right)}$$

2D zero Reynolds number flow in a wedge formed by two semi-infinite plates.