Chapter 8

The Streamfunction and Vorticity

- For 2D incompressible flows, it is possible to recast the Navier-Stokes equations in an alternative form in terms of the streamfunction and the vorticity.

- In many applications, the streamfunction-vorticity form of the Navier Stokes equations provides better insight into the physical mechanisms driving the flow than the 'primitive variable' formulation in terms of u, v and p.

- The streamfunction and vorticity formulation is also useful for numerical work since it avoids some problems resulting from the discretisation of the continuity equation.

> Unless specifically stated, all results in this chapter are restricted to 2D incompressible flows.

8.1 The Streamfunction

- The streamfunction is defined as

$$\psi_A(P) = \int_A^P \mathbf{u} \cdot \mathbf{n} \, ds, \tag{8.1}$$

where the integral has to be evaluated along a line from the arbitrary but fixed point A to point P. $\mathbf{n}$ is the unit normal on the line from A to P. We regard $\psi_A(P)$ as a function of the location of point P.
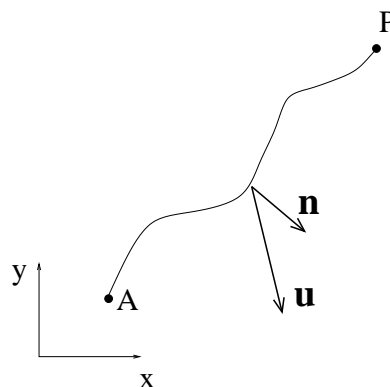

Figure 8.1: Sketch illustrating the definition of the streamfunction.

- The sketch in Fig. 8.1 shows that $\mathbf{u} \cdot \mathbf{n}$ is equal to the component of the velocity $\mathbf{u}$ that crosses the line AP. Therefore $\psi_A(P)$ represents the volume flux (per unit depth in the z-direction) through the line between A and P.

- Evaluating $\psi_A(P)$ along two different paths and invoking the integral form of the incompressibility constraint shows that $\psi_A(P)$ is path-independent, i.e. its value only depends on the locations of the points A and P.

- Changing the position of point A only changes $\psi_A(P)$ by a constant. It turns out that for all applications such changes are irrelevant. It is therefore common to suppress the explicit reference to A. Hence, we regard $\psi_A(P)$ as a function of the spatial coordinates only, i.e. $\psi_A(P) = \psi(P) = \psi(x, y)$.

- Streamlines are lines which are everywhere tangential to the velocity field, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal to the streamline. Hence the streamfunction ψ is constant along streamlines.

- Note that stationary impermeable boundaries are also characterised by $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal on the boundary. Therefore, ψ is also constant along such boundaries.

- Invoking the integral incompressibility constraint for an infinitesimally small triangle shows that ψ is related to the two cartesian velocity components u and v via

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{8.2}$$

- Similarly, in plane cylindrical polars, the velocity components are given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad \text{and} \quad u_\varphi = -\frac{\partial \psi}{\partial r}. \tag{8.3}$$

- Flows which are specified by a streamfunction automatically satisfy the continuity equation since

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0. \tag{8.4}$$

- For 2D flows, the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ only has one non-zero component (in the z-direction), i.e. $\boldsymbol{\omega} = \omega \mathbf{e}_z$ where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8.5}$$

Using the definition of the velocities in terms of the streamfunction shows that

$$\omega = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) \tag{8.6}$$

and therefore

$$\omega = -\nabla^2 \psi, \tag{8.7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2D Laplace operator.

8.2 The Streamfunction-Vorticity form of the Navier-Stokes equations

- Straightforward algebraic manipulation of the 3D momentum equations transforms them into the *vorticity transport equation*

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \tag{8.8}$$

(see the separate handout for the derivation; this equation is valid in 3D).

- This equation shows that the rate of change of the vorticity of material particles, $D\boldsymbol{\omega}/Dt$, is controlled by 'vortex stretching' (described by $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$; this is a familiar result from inviscid fluid mechanics) and by diffusion (described by $\nu\nabla^2\boldsymbol{\omega}$). The diffusion of vorticity only occurs in viscous flows.

- For 2D flows, vortex stretching is absent since $\mathbf{u} = u(x,y)\,\mathbf{e}_x + v(x,y)\,\mathbf{e}_y$ and $\boldsymbol{\omega} = \omega(x,y)\,\mathbf{e}_z$ and therefore $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0$.

- For 2D flows, the scalar vorticity transport equation

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega \tag{8.9}$$

together with the equation for the vorticity in terms of the streamfunction

$$\omega = -\nabla^2\psi \tag{8.10}$$

and

$$u = \partial\psi/\partial y \quad \text{and} \quad v = -\partial\psi/\partial x \tag{8.11}$$

provide the streamfunction-vorticity formulation of the Navier-Stokes equations. It consists of only two PDEs for the scalars ω and ψ rather than the three PDEs for u, v and p in the 'primitive variable' form.

- In the limit of zero Reynolds number, only one fourth-order PDE for the streamfunction ψ needs to be solved, namely the biharmonic equation

$$\nabla^4\psi = 0, \tag{8.12}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{8.13}$$

This can be shown by, e.g., taking the curl of the Stokes equations.