Chapter 6

Curvilinear Coordinates

- For flows in circular (or spherical) geometries, cartesian coordinates are not the most convenient coordinate system to work in.

- The transformation of the Navier-Stokes and continuity equations to other coordinate systems is straightforward (if messy) and is based on a simple coordinate transformation, such as $x = r\cos\varphi,\ y = r\sin\varphi$ for the transformation between 2D cartesian and plane cylindrical polar coordinates. Following the usual rules, we can transform differential operators to the new coordinates, e.g.
$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}. \tag{6.1}$$

- We also need to transform vectors to the new coordinate system by decomposing them into the new basis vectors, e.g.,
$$\mathbf{u} = u_x\mathbf{e}_x + u_y\mathbf{e}_y = u_r\mathbf{e}_r + u_\varphi\mathbf{e}_\varphi, \tag{6.2}$$
where u_r and u_φ are the velocity components in radial and circumferential direction.

- Note that in curvilinear coordinates, the basis vectors depend on the coordinates (e.g. $\mathbf{e}_r = (\cos\varphi, \sin\varphi)$). Hence, any differential operator acting on a vector acts on the basis vectors as well as the components themselves. The resulting vector then has to be decomposed into the basis vectors. This makes the resulting expressions considerably more complicated than their equivalents in cartesian coordinates (see the Navier Stokes equations in curvilinear coordinates in chapter 3).

- Provided we restrict ourselves to orthogonal coordinate systems (such as cylindrical and spherical polar coordinates) we can still use the index notation and the summation convention. For instance, in plane cylindrical polars the traction boundary condition can be written as
$$t_i = (-p\delta_{ij} + 2\mu e_{ij})n_j \quad \text{where } i,j \text{ represent the coordinate directions } r \text{ and } \varphi. \tag{6.3}$$

- Thus, for instance, the r-component of the traction $\mathbf{t} = t_r\mathbf{e}_r + t_\varphi\mathbf{e}_\varphi$ is given by
$$t_r = -pn_r + 2\mu(e_{rr}n_r + e_{r\varphi}n_\varphi) \tag{6.4}$$
where $\mathbf{n} = n_y\mathbf{e}_r + n_\varphi\mathbf{e}_\varphi$ is the outer unit normal on the fluid, decomposed into the cylindrical basis vectors and the e_{ij} are the components of the rate of strain tensor in plane cylindrical polars as given in chapter 3.