Chapter 3

Stress, Cauchy's equation and the Navier-Stokes equations

3.1 The concept of traction/stress

- Consider the volume of fluid shown in the left half of Fig. 3.1. The volume of fluid is subjected to distributed external forces (e.g. shear stresses, pressures etc.). Let $\Delta\mathcal{F}$ be the resultant force acting on a small surface element ΔS with outer unit normal $\mathbf{n}$, then the traction vector $\mathbf{t}$ is defined as:

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{\mathcal{F}}}{\Delta S} \tag{3.1}$$

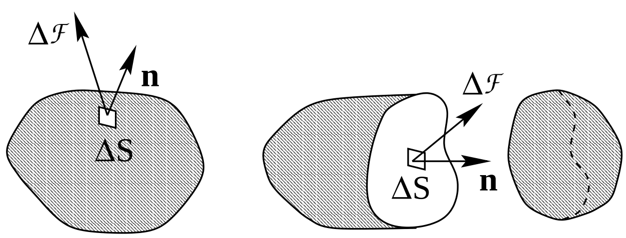

Figure 3.1: Sketch illustrating traction and stress.

- The right half of Fig. 3.1 illustrates the concept of an (internal) stress $\mathbf{t}$ which represents the traction exerted by one half of the fluid volume onto the other half across a ficticious cut (along a plane with outer unit normal $\mathbf{n}$) through the volume.

3.2 The stress tensor

- The stress vector $\mathbf{t}$ depends on the spatial position in the body and on the orientation of the plane (characterised by its outer unit normal $\mathbf{n}$) along which the volume of fluid is cut:

$$t_i = \tau_{ij} n_j, \tag{3.2}$$

where $\tau_{ij} = \tau_{ji}$ is the symmetric *stress tensor*.

- On an infinitesimal block of fluid whose faces are parallel to the axes, the component τ_{ij} of the stress tensor represents the traction component in the positive i-direction on the face $x_j = const.$ whose outer normal points in the positive j-direction (see Fig. 3.2).

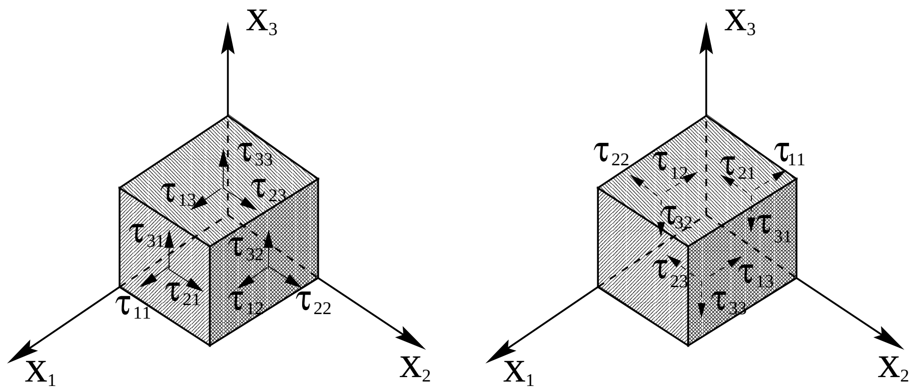

Figure 3.2: Sketch illustrating the components of the stress tensor.

3.3 Examples for simple stress states

- Hydrostatic pressure: $\tau_{ij} = -P_0 \; \delta_{ij}$; note that $t_i = \tau_{ij} n_j = -P_0 \; \delta_{ij} n_j = -P_0 \; n_i$, i.e. the stress on any surface is normal to the surface and 'presses against it' (i.e. acts in the direction opposite to the outer normal vector) which is precisely what we expect a pure pressure to do; see left half of Fig. 3.3
- Pure shear stress: E.g. $\tau_{12} = \tau_{21} = T_0$, $\tau_{ij} = 0$ otherwise; see right half of Fig. 3.3. This sketch also illustrates that the symmetry of the stress tensor is related to the balance of moments: If τ_{21} were not equal to τ_{12} (i.e. if the tangential stress acting on the vertical faces was not equal to the tangential stress acting on the horizontal ones) then the block would rotate about the x_3 axis.

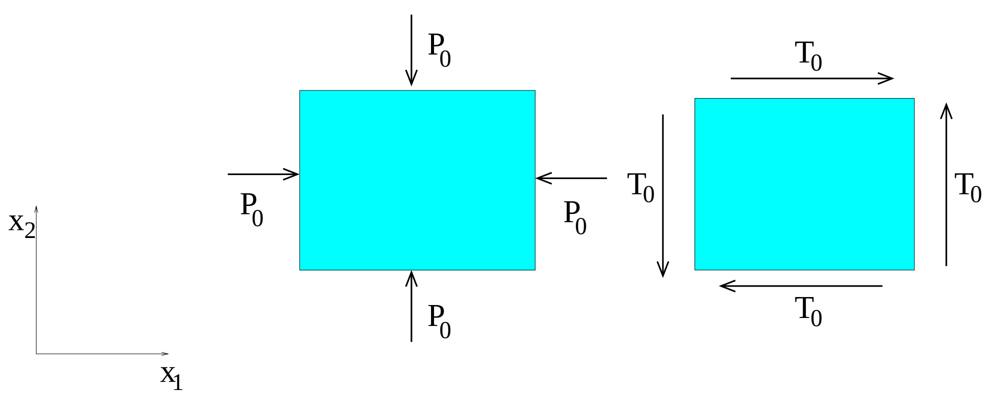

Figure 3.3: Simple stress states: Hydrostatic pressure (left) and pure shear stress (right).

3.4 Cauchy's equation

- Cauchy's equation is obtained by considering the equation of motion ('sum of all forces = mass times acceleration') of an infinitesimal volume of fluid. For a fluid which is subject to a body force (a force per unit mass) F_i, Cauchy's equation is given by

$$\rho a_i = \rho F_i + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{3.3}$$

where ρ is the density of the fluid. a_i is the acceleration of the fluid, given by (2.5), therefore Cauchy's equation can also be written as

$$\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial \tau_{ij}}{\partial x_j} \tag{3.4}$$

or

$$\rho\left(\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}\right)=\rho F_i+\frac{\partial \tau_{ij}}{\partial x_j}. \tag{3.5}$$

- Note that Cauchy's equation is valid for *any* continuum (not just fluids!) provided its deformation is described by an Eulerian approach.

3.5 The constitutive equations for a Newtonian incompressible fluid

- In chapter 2 we derived a quantity (the rate of strain tensor ϵ_{ij}) which provides a mathematical description of the rate of deformation of the fluid.
- Cauchy's equation provides the equations of motion for the fluid, provided we know what state of stress (characterised by the stress tensor τ_{ij}) the fluid is in.
- The constitutive equations provide the missing link between the rate of deformation and the resulting stresses in the fluid.
- A large number of practically important fluids (e.g. water and oil) are incompressible and exhibit a linear relation between the shear rate of strain and the shear stresses. These fluids are known as 'Newtonian Fluids' and their constitutive equation is given by

$$\tau_{ij}=-p\delta_{ij}+2\mu\epsilon_{ij}, \tag{3.6}$$

or, using the definition of the rate of strain tensor,

$$\tau_{ij}=-p\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{3.7}$$

where p is the pressure in the fluid and μ is the 'dynamic viscosity', a quantity that has to be determined experimentally.

- Note that there are also many fluids which do not behave as Newtonian fluids and have different constitutive equations (e.g. toothpaste, mayonaise). Not very imaginatively, these are often called 'Non-Newtonian Fluids' – the behaviour of these fluids is covered in a different lecture.

3.6 The Navier-Stokes equations for incompressible Newtonian fluids

- We insert the constitutive equations for an incompressible Newtonian fluid into Cauchy's equations and obtain the famous Navier-Stokes equations

$$\rho\left(\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}\right)=\rho F_i-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j^2}, \tag{3.8}$$

or symbolically

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=\rho\mathbf{F}-\nabla p+\mu\nabla^2\mathbf{u}. \tag{3.9}$$

Dividing the momentum equations by ρ provides an alternative form

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=F_i-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j^2}, \tag{3.10}$$

where $\nu=\mu/\rho$ is the 'kinematic viscosity'.

- In combination with the equation of continuity

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{3.11}$$

  or symbolically

$$\nabla \cdot \mathbf{u} = 0, \tag{3.12}$$

  the three momentum equations form a system of four coupled nonlinear, partial differential equations of parabolic type (second order in space and first order in time) for the three velocity components u_i and the pressure p.

The governing equations in selected coordinate systems

Rectangular cartesian coordinates

The rate of strain tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}$$

where

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \qquad \epsilon_{xy} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\epsilon_{yz} = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \qquad \epsilon_{zx} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]$$

The vorticity

$$\omega = \operatorname{curl}\mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

The Navier Stokes equations

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 u, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 v, \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w,
\end{aligned}$$

$$\operatorname{div}\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The Laplace operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Cylindrical Polar Coordinates

Relation to Cartesian coordinates:

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi, \\ z &= z \end{aligned}$$

Velocity components:

$$u = u_r, \qquad v = u_\varphi, \qquad w = u_z$$

The rate of strain tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\varphi} & \epsilon_{rz} \\ \epsilon_{\varphi r} & \epsilon_{\varphi\varphi} & \epsilon_{\varphi z} \\ \epsilon_{zr} & \epsilon_{z\varphi} & \epsilon_{zz} \end{pmatrix}$$

where

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \qquad \epsilon_{\varphi\varphi} = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \qquad \epsilon_{r\varphi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \varphi}\right]$$

$$\epsilon_{\varphi z} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial w}{\partial \varphi} + \frac{\partial v}{\partial z}\right] \qquad \epsilon_{rz} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]$$

The vorticity

$$\omega = \operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial w}{\partial \varphi} - \frac{\partial v}{\partial z},\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial \varphi}\right).$$

The Navier Stokes equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \varphi} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} &= -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi}\right], \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + w\frac{\partial v}{\partial z} + \frac{uv}{r} &= -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi}\right], \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w, \end{aligned}$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0.$$

The Laplace operator

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

Spherical Polar Coordinates

Relation to Cartesian coordinates:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta\cos\varphi, \\ z &= r\sin\theta\sin\varphi \end{aligned}$$

Velocity components:

$$u = u_r, \qquad v = u_\theta, \qquad w = u_\varphi$$

The rate of strain tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{r\varphi} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta\varphi} \\ \epsilon_{\varphi r} & \epsilon_{\varphi\theta} & \epsilon_{\varphi\varphi} \end{pmatrix}$$

where

$$\epsilon_{rr} = \frac{\partial u}{\partial r} \qquad \epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r}$$

$$\epsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi} + \frac{u}{r} + \frac{v\cot\theta}{r} \qquad \epsilon_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\theta}\right]$$

$$\epsilon_{\theta\varphi} = \frac{1}{2}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{w}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v}{\partial\varphi}\right] \qquad \epsilon_{\varphi r} = \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi} + r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right]$$

The vorticity

$$\omega = \operatorname{curl}\mathbf{u} = \left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(w\sin\theta) - \frac{\partial v}{\partial\varphi}\right],\ \ \frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rw),\ \ \frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial\theta}\right).$$

The Navier Stokes equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial\varphi} - \frac{v^2+w^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta} - \frac{2v\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial w}{\partial\varphi}\right],$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial\varphi} + \frac{uv}{r} - \frac{w^2\cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial\theta} + \nu\left[\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial\theta} - \frac{v}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial w}{\partial\varphi}\right],$$

$$\begin{aligned} &\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial\varphi} + \frac{uw}{r} - \frac{vw\cot\theta}{r} = \\ &\qquad -\frac{1}{\rho r\sin\theta}\frac{\partial P}{\partial\varphi} + \nu\left[\nabla^2 w - \frac{w}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v}{\partial\varphi}\right], \end{aligned}$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi} = 0.$$

The Laplace operator

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$