Chapter 2

The Kinematics of Fluid Flow

2.1 The Eulerian flow field

- Eulerian description of the flow field: The velocity $\mathbf{u}$ is given as a function of the position relative to a spatially fixed coordinate system $(x, y, z) = (x_1, x_2, x_3) = x_i$, and of time t.

$$\mathbf{u} = \mathbf{u}(x, y, z, t) \quad \text{or in index notation:} \quad u_i = u_i(x_j, t). \tag{2.1}$$

- Note that at different times, different material particles will be at a given spatial position. The particle paths (i.e. the trajectories $x_i^p(t)$ of individual material particles which are at position $x_i^{(0)}$ at time $t = t_0$) are obtained by integrating

$$\frac{\partial x_i^p(t)}{\partial t} = u_i(x_j^p, t) \tag{2.2}$$

subject to the initial conditions

$$x_i^p(t = 0) = x_i^{(0)}. \tag{2.3}$$

2.2 The material derivative

- The acceleration a_i of the material particle that is at position x_j at time t is given by

$$a_i(x_j, t) = \left(\frac{d}{dt} \; u_i(x_j^p(t), t) \right) \bigg|_{x_j^p(t) = x_j} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_k} \frac{\partial x_k^p(t)}{\partial t}. \tag{2.4}$$

Comparing this to (2.2) shows that this can be written as

$$a_i = \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \quad \text{or symbolically} \quad \mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{2.5}$$

- The differential operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k} \quad \text{or symbolically} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \tag{2.6}$$

is known as the 'material (or substantial) derivative'. Given any function $\phi(x_j, t)$, $D\phi/Dt$ represents the rate of change of ϕ experienced by an observer travelling with the velocity $u_i(x_j, t)$.

2.3 Vorticity and the rate of strain tensor

- The velocity field can be decomposed into four fundamental 'modes' which correspond to the translation, rotation, shearing and dilation of small material elements contained in the flow. The velocity in the vicinity of a certain point x_k can be expressed as

$$u_i(x_k + \delta x_k) = \underbrace{u_i(x_k)}_{\text{rigid body translation}} + \underbrace{\omega_{ij}\ \delta x_j}_{\text{rigid body rotation}} + \underbrace{\epsilon_{ij}\ \delta x_j}_{\text{shearing and dilation}} , \tag{2.7}$$

where ω_{ij} is the antisymmetric rate of rotation tensor

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.8}$$

and ϵ_{ij} is the symmetric rate of strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.9}$$

- The first term in (2.7) represents a rigid body translation: If $\epsilon_{ij} = \omega_{ij} = 0$ then all particles have the same velocity, i.e. the fluid moves in a straight line as a rigid body.
- The physical meaning of the second term in (2.7) is revealed by rewriting $\omega_{ij}\ \delta x_j$ symbolically as a cross product in the form $\mathbf{\Omega} \times \delta\boldsymbol{x}$ where $\mathbf{\Omega} = (\omega_{32}, \omega_{13}, \omega_{21})$ is the rate of rotation vector.

  This is illustrated in Fig. 2.1: The differential velocity $\delta\mathbf{u} = \mathbf{u}(x_j) - \mathbf{u}(x_j + \delta x_j)$ induced by a rigid body rotation about point P with rotation rate $\mathbf{\Omega}$ is given by $\delta\mathbf{u} = \mathbf{\Omega} \times \delta\boldsymbol{x}$.

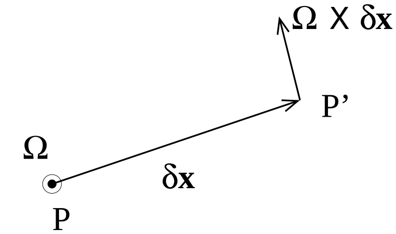

Figure 2.1: Sketch illustrating the motion induced by a rigid body rotation about point P with rotation rate $\mathbf{\Omega}$. In this sketch the rate of rotation vector $\mathbf{\Omega}$ points vertically out of the paper.

The rotation rate $\mathbf{\Omega}$ is equal to half the *vorticity* $\boldsymbol{\omega}$, i.e.

$$2\,\mathbf{\Omega} = \boldsymbol{\omega} = \text{curl}\ \ \mathbf{u} = \nabla \times \mathbf{u} = \begin{pmatrix} (\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) \\ (\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) \\ (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) \end{pmatrix} \tag{2.10}$$

- The diagonal entries of the rate of strain tensor ϵ_{ij} represent the extensional rate of strain in the direction of the three cartesian coordinate axes, as illustrated in Fig. 2.2, e.g. $Ds_1/Dt = e_{11} = \partial u_1/\partial x_1$

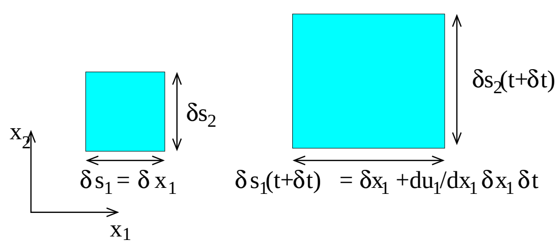

Figure 2.2: A rectangular block of fluid undergoes a purely extensional deformation which changes the lengths of the material lines parallel to the coordinate axes.

- The off-diagonal entries of the rate of strain tensor ϵ_{ij} represent the shear rate of strain (in fact, they are equal to half the shear rate in the appropriate directions; see Fig. 2.3).

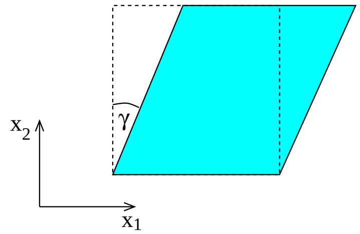

Figure 2.3: Sketch illustrating the shearing of an initially rectangular block of fluid at a rate $D\gamma/Dt = 2\ e_{12} = (\partial u_1/\partial x_2 + \partial u_2/\partial x_1)$.

2.4 The equation of continuity

- Mass conservation requires that the rate at which mass is transported over the surface ∂V of a spatially fixed volume V must be equal to the rate of change of mass in this volume. This physical statement can be formulated in an integral or a differential form:

- The integral form of the equation of continuity is given by

$$\int_V \frac{d\rho}{dt}\, dV + \oint_{\partial V} \rho u_i n_i\ dS = 0, \tag{2.11}$$

or in symbolic form

$$\int_V \frac{d\rho}{dt}\, dV + \oint_{\partial V} \rho \mathbf{u} \cdot \mathbf{n}\ dS = 0, \tag{2.12}$$

where ρ is the density of the fluid (i.e. the mass per unit volume), and $\mathbf{n}$ is the outer unit normal on the surface ∂V of the spatially fixed volume V (note that $\mathbf{u} \cdot \mathbf{n} < 0$ corresponds to an inflow).

- The corresponding differential form of the equation of continuity can be derived by applying the integral statement to an infinitesimally small block of fluid. The result is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0. \tag{2.13}$$

Using the material derivative introduced in (2.6), this expression can be rewritten as

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} = 0. \tag{2.14}$$

- The latter equation shows that for incompressible fluids (i.e. fluids for which the density of material fluid elements is constant and thus $D\rho/Dt = 0$), the equation of continuity presents a purely kinematic constraint on the velocity field, namely

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.15}$$

or in symbolic form

$$\text{div}\ \mathbf{u} = 0 \qquad \text{or} \qquad \nabla \cdot \mathbf{u} = 0. \tag{2.16}$$