Where have we (you!) seen $x = x_P + x_H$ before?

Recall:

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + \alpha\, y_1(x) + \beta\, y_2(x),$$

where:

- α and β are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Compare this to the solution of the system of linear (algebraic) equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{b}$ a given vector of size n.

The general solution $\mathbf{x}$ (another vector of size n) is given by

$$\mathbf{x} = \mathbf{x}_P + \mathbf{x}_H$$

where

- $\mathbf{x}_P$ is a(ny) particular solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$
- $\mathbf{x}_H$ is the *general* solution of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.

Example

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Note that the matrix is singular, so $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions!

- Transform into "triangular" form

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

  showing that the RHS is consistent. We're left with one equation for three unknowns.

- Set $x_2 = \alpha$ and $x_3 = \beta$, where α and β are arbitrary constants.
- The general solution is: $x_1 = 1 + \alpha$ and, of course, $x_2 = \alpha$ and $x_3 = \beta$.
- Rewrite in vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{x}_P} + \underbrace{\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{x}_H}$$

- Note that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 3.1415 \end{pmatrix}}_{\mathbf{x}'_P} + \underbrace{\alpha' \begin{pmatrix} -42.2 \\ -42.2 \\ 1145.2 \end{pmatrix} + \beta' \begin{pmatrix} 523.2 \\ 523.2 \\ 13.423 \end{pmatrix}}_{\mathbf{x}'_H}$$

  is another (not so pretty) representation of the general solution.

The key features of both solutions are:

- $\mathbf{x}_P$ and $\mathbf{x}'_P$ solve the inhomogeneous equation.
- $\mathbf{x}_H$ and $\mathbf{x}'_H$ "span the null space" of $\mathbf{A}$, i.e. they
    1. satisfy $\mathbf{A}\mathbf{x} = \mathbf{0}$,
    2. are nonzero,
    3. are linearly independent.

"Off the record comment":

In linear algebra it's "easier" to overlook the additional solutions represented by $\mathbf{x}_H$. In an ODE context, the fact that BCs [or ICs] have to be satisfied too, tends to provide an instant "reminder" that just having *a* particular solution of the ODE is not enough to solve the entire IVP/BVP.

Summary:

Tasks for soln. of 2nd order linear ODEs:

① find 2 lin. indep. nonzero solns. of homog. ODE

② find a(ny) particular soln of the inhomog. ODE

③ Add & apply ICs.

Constant coefficient ODEs

$$\boxed{\begin{gathered} y'' + p y' + q y = r(x) \\ p \;\&\; q \text{ are constants} \end{gathered}}$$

I) Soln. of homog. ODE

$$y'' + p y' + q y = 0 \quad (H)$$

(Note: soln exists $\forall x$)

Idea: Try to find a soln. that has the same x-dependence for each term in (H)

Ansatz:
$$y = \cancel{Ae^{\lambda x}} \; A e^{\lambda x}$$

into ODE
$$y' = A \lambda e^{\lambda x}$$
$$y'' = A \lambda^2 e^{\lambda x}$$

$$A\left(\lambda^2 + p\lambda + q\right) e^{\lambda x} = 0 \quad \forall x$$

can achieve this by
$A = 0 \Rightarrow y \equiv 0$ (not helpfu...

So choose λ s.th.

$$\lambda^2 + p\lambda + q = 0$$

"characteristic polynomial"
is satisfied for

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$\Rightarrow$ Two nonzero solns of (H) are

$$y_1(x) = e^{\lambda_1 x} \quad \& \quad y_2(x) = e^{\lambda_2 x}$$

<u>But</u>: possibility of repeated & complex roots!

$\Rightarrow$ 3 cases

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① $p^2 > 4q$: λ_1 & λ_2 are real & distinct

In that case the gen. soln. of (H)

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

② $p^2 = 4q$: Repeated roots

$$\lambda_1 = \lambda_2 = \lambda = -\frac{p}{2}$$

Our ansatz now only produces one soln, $y_1(x) = e^{\lambda x}$

However: $y_2(x) = x e^{\lambda x}$ is another, lin. indep. soln.

Proof: Note: $p = -2\lambda$

$$q = \frac{1}{4}p^2 = \lambda^2$$

$$y' = e^{\lambda x}(1 + \lambda x)$$

$$y'' = \lambda(2 + \lambda x)e^{\lambda x}$$

into ODE:

$$y'' + py' + qy \overset{?}{=} 0$$

$$e^{\lambda x}\Big(\underbrace{\lambda(2+\lambda x)}_{y''} + \underbrace{p}_{-2\lambda}\underbrace{(1+\lambda x)}_{y'} + \underbrace{q}_{\lambda^2}\underbrace{x}_{y}\Big) \overset{?}{=} 0$$

$$\cancel{2\lambda} + \lambda^2 x - \cancel{2\lambda} - 2\lambda^2 x + \lambda^2 x \overset{?}{=} 0$$

✓

So in this case the gen. soln. of (H) is

$$y(x) = Ae^{\lambda x} + Bxe^{\lambda x}$$

③ $p^2 < 4q$: λ_{12} are complex conjugates

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$= -\frac{p}{2} \pm i\sqrt{q - \left(\frac{p}{2}\right)^2}$$

$$= \mu \pm i\omega$$

$$y(x) = \hat{A} e^{(\mu+i\omega)x} + \hat{B} e^{(\mu-i\omega)x}$$

$$= e^{\mu x}\left(\hat{A} e^{i\omega x} + \hat{B} e^{-i\omega x}\right)$$

complex

If we want a real soln then $\hat{A}$ & $\hat{B}$ must be complex too!

$$e^{\pm i\omega x} = \cos(\omega x) \pm i \sin(\omega x)$$

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We can replace the complex exponentials by $\sin(\omega x)$ & $\cos(\omega x)$.

(EXERCISE)

So the gen. real soln of (*) is

$$y(x) = e^{\mu x}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

Example:

$$y'' - 3y' + 2y = 0$$

$$y \sim e^{\lambda x}$$

$$e^{\lambda x}\underbrace{\left(\lambda^2 - 3\lambda + 2\right)}_{=0} \overset{!}{=} 0 \quad \forall x$$

$$\lambda_{1,2} = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2}$$

$$= \frac{3}{2} \pm \sqrt{\frac{9-8}{4}} = \frac{3}{2} \pm \frac{1}{2}$$

$\lambda_1 = 2, \ \lambda_2 = 1$

$$y = Ae^{2x} + Be^{x}$$

Example:

$$y'' + 2y' + y = 0$$

char. poly:

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0$$

$\lambda_{12} = -1$ repeated root!

$$y = Ae^{-x} + Bxe^{-x}$$

Example:

$$y'' + 2y' + 5y = 0$$

char. poly:

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_{12} = -\frac{2}{2} \pm \sqrt{\left(\frac{2}{2}\right)^2 - 5}$$

$$\lambda_{12} = -1 \pm 2i$$

$$= \mu \pm i\omega$$

$$y(x) = e^{-x}\left(\hat{A}e^{2ix} + \hat{B}e^{-2ix}\right)$$

$$= e^{-x}\left(A\cos(2x) + B\sin(2x)\right)$$

(II) Particular solns

$$y'' + p y' + q y = r(x)$$

Gen. soln:

$$y(x) = y_p(x) + \underbrace{A y_1(x) + B y_2(x)}_{\checkmark}$$

Strategy: Trial & error method guided by the form of $r(x)$.

$\Rightarrow$ "Method of undetermined coefficients"

To illustrate the idea & the pitfalls consider:

$$y'' + p y' + q y = A e^{ax}$$

Note: A & a are given.

Given the form of the RHS, try:

$$y = Ce^{ax}$$

into ODE:

$$y'' + py' + qy = Ae^{ax}$$

$$\cancel{Ce^{ax}}\left(a^2 + pa + q\right) = A\cancel{e^{ax}}$$

So

$$C = \frac{A}{a^2 + pa + q}$$

So, a soln. of the ODE is

$$y_p(x) = \frac{A}{a^2 + pa + q}\, e^{ax}$$