1st order linear ODEs

$$y' + p(x)\,y = q(x)$$

$$y' \underbrace{- x}_{p(x)}\, y = \underbrace{x}_{q(x)}$$

$$y(x) = -1 + c\,\exp\left(\frac{x^2}{2}\right)$$

Note: Soln. has two parts

$$y = y_P + y_H$$

y_P = a(ny) particular soln of the full ODE

y_H = general soln. of the homogeneous ODE, i.e. the ODE for $q(x) = 0$

In our example: (2

- $y_P = -1$ does it solve the full ODE?

$y_P' = 0$

into ODE:

$$y_P' - x y_P \stackrel{?}{=} x$$

$$(-x)(-1) \stackrel{?}{=} x \quad \checkmark$$

- $y_H = C \exp\left(\frac{x^2}{2}\right)$ is this the most general soln. of homog. ODE

$$y_H' - x y_H \stackrel{?}{=} 0$$

$$\frac{dy_H}{dx} - x y_H = 0$$

$$\int \frac{1}{y_H} \, dy_H = \int x \, dx$$

$$\ln|y_H| = \frac{1}{2}x^2 + A$$

$$y_H = \exp\left(\frac{1}{2}x^2 + A\right)$$

$$y_H = C \exp\left(\frac{1}{2}x^2\right)$$

This structure of the soln. is always the same for linear ODEs. 3

§4 2^nd order ODEs

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$$\boxed{\begin{array}{l} F(x, y, y', y'') = 0 \\ + 2\,BC \text{ or } 2\,ICs \end{array}}$$

(slides)

Some theory for *linear* 2nd order ODEs

Existence and Uniqueness

Consider the *linear* second-order ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x), \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the functions $p(x)$, $q(x)$ and $r(x)$ are given.

Theorem

> If the functions $p(x)$, $q(x)$ and $r(x)$ are continuous functions of x in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in the entire interval I.

Notes:

- This is the promised extension of the statement for first-order problems. The extension to even higher-order linear ODEs should be obvious...
- If the functions $p(x)$, $q(x)$ and $r(x)$ are "well-behaved" (no jumps, singularities, etc.), the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

The homogeneous ODE & superposition of its solutions

If we set $r(x) = 0$ in the *inhomogeneous* ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x), \qquad \text{(I)}$$

we obtain the corresponding *homogeneous* ODE

$$y'' + p(x)\,y' + q(x)\,y = 0. \qquad \text{(H)}$$

A trivial (?) but useful observation

If $y_1(x)$ and $y_2(x)$ are two solutions of (H) then the linear combination

$$y_3(x) = A\,y_1(x) + B\,y_2(x)$$

is also a solution, for any values of the constants A and B.

Linear independence

To see why this is a useful observation, we need to define the concept of linear independence: Two nonzero functions $y_1(x)$ and $y_2(x)$ are linearly independent if

$$A\,y_1(x) + B\,y_2(x) = 0 \quad \forall x \qquad \Longleftrightarrow \qquad A \equiv B \equiv 0$$

(...just as in linear algebra...).

Examples:

- $y_1(x) = x$ and $y_2(x) = 3\,x^2$ are linearly independent.
- $y_1(x) = x$ and $y_2(x) = 3\,x$ are linearly dependent – they're just multiples of each other.

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$y'' + py' + qy = 0 \quad (H)$

Claim: if y_1 & y_2 are solns of (H) then $Ay_1 + By_2$ is also a soln.

Proof:

$$y = Ay_1 + By_2$$
$$y' = Ay_1' + By_2'$$
$$y'' = Ay_1'' + By_2''$$

into ODE:

$$Ay_1'' + By_2'' + pAy_1' + pBy_2' + qAy_1 + qBy_2 \overset{?}{=} 0$$

$$A\underbrace{(y_1'' + py_1' + qy_1)}_{0} + B\overbrace{(y_2'' + py_2' + qy_2)}^{0} \overset{?}{=} 0$$

✓

Lin. indep:

$$y_1 = x \qquad y_2 = 3x^2$$

$$A y_1 + B y_2 = 0 \qquad \forall x$$

$$A x + 3B x^2 = 0 \qquad \forall x$$

check for

$x = 1: \quad A + 3B = 0$

$x = -1: \quad -A + 3B = 0$

$$6B = 0 \Rightarrow B = 0$$

$$A = 0$$

$\Rightarrow$ lin. indep!

$$y_1 = x \qquad y_2 = 3x$$

$$A y_1 + B y_2 = 0 \qquad \forall x$$

$$A x + 3B x = 0 \qquad \forall x$$

$$x (A + 3B) = 0 \qquad \forall x$$

$$A = -3B \qquad B = 1, \; A = -3$$

$\Rightarrow$ lin. dependent!

Fundamental solutions of the homogeneous ODE

Theorem

Any solution of the homogeneous ODE

$$y'' + p(x)\, y' + q(x)\, y = 0. \qquad \text{(H)}$$

can be written as a linear combination of *any* two non-zero, linearly independent solutions, $y_1(x)$ and $y_2(x)$, say:

$$y(x) = A\, y_1(x) + B\, y_2(x).$$

The two non-zero, linearly independent solutions $\{y_1(x),\, y_2(x)\}$ are called "fundamental solutions" of the homogeneous ODE (H).

Notes:

- The set of fundamental solutions is not unique!

The general solution of the inhomogeneous ODE

Theorem

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + A\,y_1(x) + B\,y_2(x),$$

where:

- A and B are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Notes:

- Note the similarities between the structure of the solution of the linear ODE and the structure of the solution of the linear (algebraic) equation $\mathbf{Ax} = \mathbf{b}$. This is not accidental! There are deep connections between the two fields – matrices and the homogeneous part of a linear ODE are both "linear operators".
- The values of the constants A and B are determined by the boundary or initial conditions.

Example:

$$y'' + \underbrace{\frac{1}{x}}_{p(x)} y' \underbrace{- \frac{1}{x^2}}_{q(x)} y = \underbrace{-\frac{1}{x^2}}_{r(x)}$$

<u>IC:</u> $y(x=1) = 1$

$y'(x=1) = 1$

$\bar{x} = 1, \quad \bar{y} = 1, \quad z = 1$

$$\left.\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \\ r(x) &= -\frac{1}{x^2} \end{aligned}\right\} \begin{aligned} &p, q, r \text{ are cont. fcts of } x \\ &\text{in } I_1 = \mathbb{R}^+ \\ &\quad\;\; I_2 = \mathbb{R}^- \end{aligned}$$

$$\bar{x} \in I_1$$

$\Rightarrow$ There exists a unique soln of IVP in I_1.

• y_P:

$$y_p = 1$$
$$y_p' = 0$$
$$y_p'' = 0$$

$$\cancel{y_p''} + \cancel{\frac{1}{x} y_p'} - \frac{1}{x^2} \underset{1}{y_p} \stackrel{?}{=} -\frac{1}{x^2}$$

✓

• y_H:

$y_1 = x$ ✓

$y_2 = \frac{1}{x}$ ✓ EXERCISE

$$y_1 = x$$
$$y_1' = 1$$
$$y_1'' = 0$$

homog.!

$$\cancel{y_1''} + \frac{1}{x} y_1' - \frac{1}{x^2} y_1 \stackrel{?}{=} 0$$

$$\frac{1}{x} 1 - \frac{1}{x^2} x \stackrel{?}{=} 0$$ ✓

y_1 & y_2 solve (H)
& are nonzero

Are they lin. indep:

$$A y_1 + B y_2 = 0 \quad \forall x$$

$$A x + B \frac{1}{x} = 0 \quad \forall x$$

$x=1$: $A + B = 0$

$x=2$ $2A + \frac{1}{2} B = 0$

$-4A - B = 0$

$\Rightarrow$ $-3A = 0$ $\Rightarrow$ $A = 0$

$\Downarrow$

$B = 0$

y_1 & y_2 are nonzero, lin. indep. solns of (H).

$\{x, \frac{1}{x}\}$ are fundamental solns. of (H)

Gen. soln. is:

$$y(x) = \underbrace{1}_{y_p} + A\underbrace{x}_{y_1} + B\underbrace{\frac{1}{x}}_{y_2}$$

Apply ICs to find the unique soln of IVP

$$\left.\begin{aligned} y(x=1) &= 1 \\ y'(x=1) &= 1 \end{aligned}\right\} \quad \ldots \quad \begin{aligned} A &= \tfrac{1}{2} \\ B &= -\tfrac{1}{2} \end{aligned}$$

(exercise)

$$\underline{\underline{y(x) = 1 + \frac{1}{2}x - \frac{1}{2x}}}$$

<u>Note:</u> soln does blow up @ $x = 0$.

<u>Q:</u> How can we obtain a unique soln even though y_p, y_1 & y_2 were not unique.

Redo with:

$$y_p = 1 + x$$

(check as exercise!) it solves the full ODE

$$\left.\begin{aligned} y_1 &= x \\ y_2 &= x + \frac{2}{x} \end{aligned}\right\}$$

EXERCISE: y_1 & y_2 are lin. indep nonzero solns of (H)

Gen. soln is

$$y = \underbrace{1+x}_{y_p} + C\underbrace{x}_{y_1} + D\underbrace{\left(x+\frac{2}{x}\right)}_{y_2}$$

apply ICs to this form of the soln

$$\left.\begin{aligned} y(x=1) &= 1 \\ y'(x=1) &= 1 \end{aligned}\right\} \ldots \quad c = D = -\frac{1}{4}$$

EXERCISE

$$y(x) = \underbrace{1+x}_{y_p} - \frac{1}{4}\underbrace{x}_{y_1} - \frac{1}{4}\underbrace{\left(x + \frac{2}{x}\right)}_{y_2}$$

$$= 1 + \frac{1}{2}x - \frac{1}{2x} \quad \text{as before.}$$

Note: E & U theorem guaranteed E & U of the soln for $x > 0$. Indeed here soln has a singularity at $x = 0$

But the E&U theorem does not say that there <u>will</u> be a singularity at $x=0$. Depends on I.C.

For example:

$$\left.\begin{aligned} y(x=1) &= \frac{3}{2} \\ y'(x=1) &= \frac{1}{2} \end{aligned}\right\} \ldots \qquad \begin{aligned} A &= \frac{1}{2} \\ B &= 0 \end{aligned}$$

$$\Rightarrow \quad y = 1 + \frac{1}{2}x$$

This actually exists for all values of $x \in \mathbb{R}$.