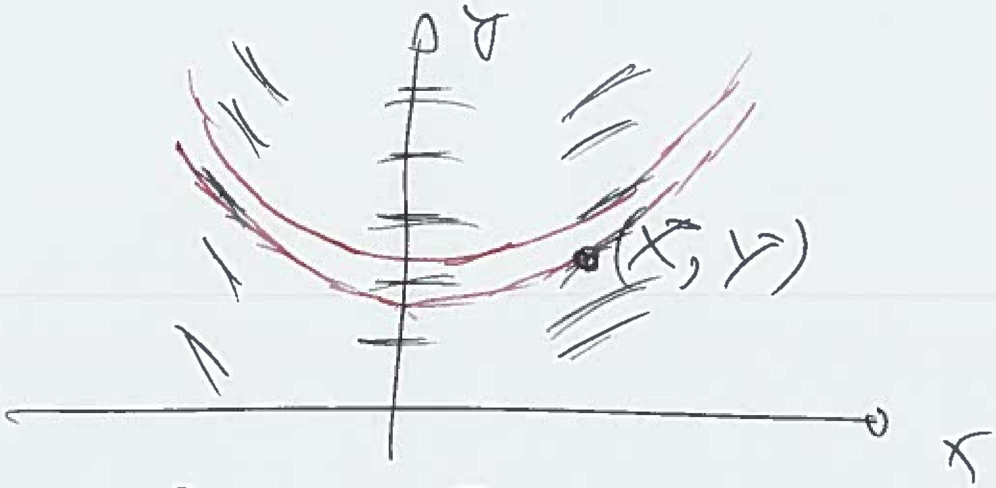
Graphical solns

$$\frac{dy}{dx} = f(x,y)$$

$\uparrow$ slope of $y(x)$



Observation: Sketching of the solns. is facilitated by identifying so-called isoclines

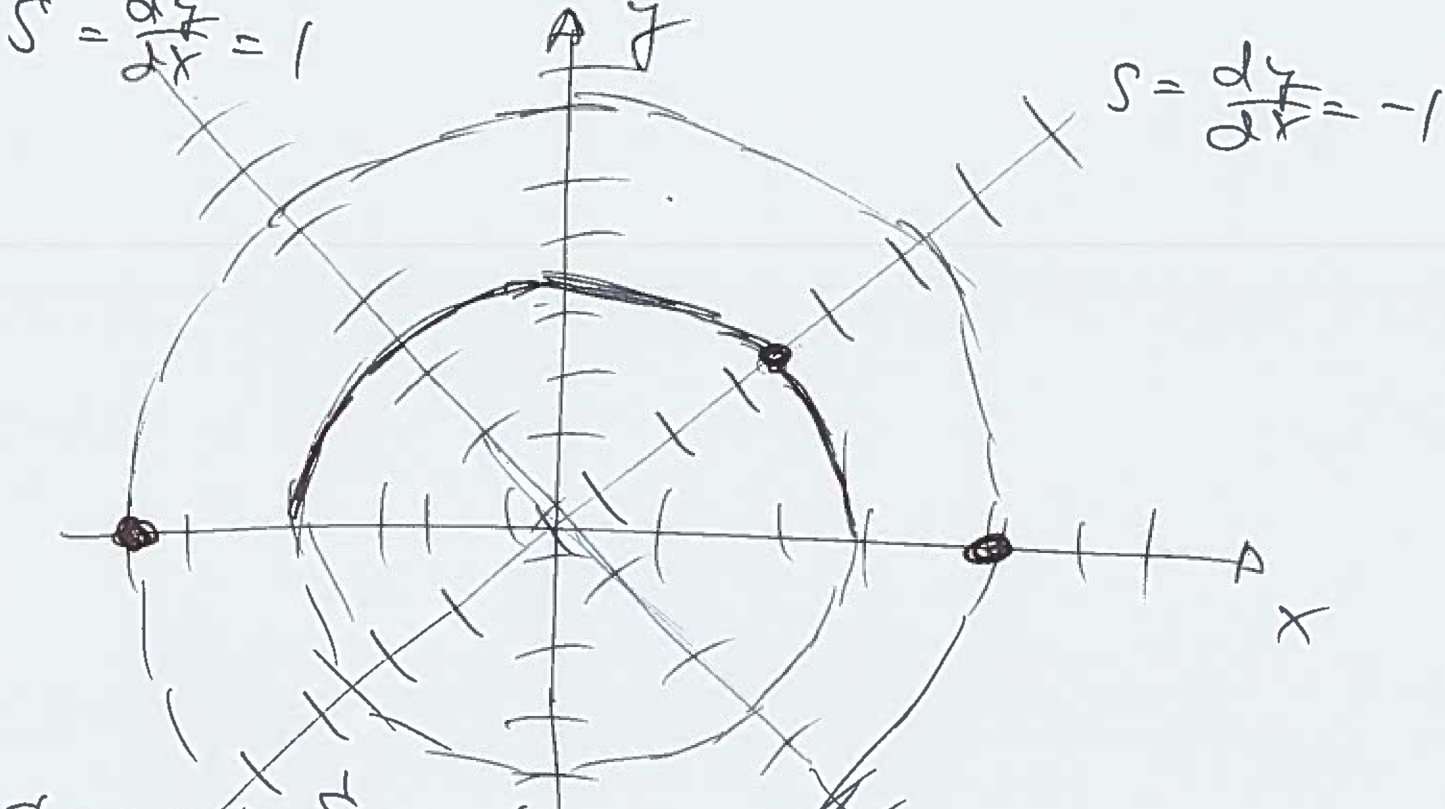
Def: Isoclines are lines in the x-y plane where $f(x,y) = \text{const}$.

Example:

$$\frac{dy}{dx} = \underbrace{-\frac{x}{y}}_{f(x,y)}$$

Isoclines are defined by

$$\boxed{-\frac{x}{y} = s = \text{const}}$$



slope = $s = -1$: $y = x$

slope = $s = +1$: $y = -x$

slope = $s = 0$: $x = 0$

as $y \to 0$: $\frac{dy}{dx} \to \infty$

Sketch suggests: soln = circular arcs

$$y = \pm\sqrt{R^2 - x^2}$$ ✓ by inspection

E & U: From graphical soln:

- uniqueness ✓ ($y \neq 0$)
- existence ✓ but only for a limited range of x
- what about $y = 0$

Recall E & U theorem req'd $f(x,y)$ to be a cont. fct. of x & y @ (x, y)

$$f(x,y) = -\frac{x}{y}$$

Condition is violated for $y = 0$. Indeed here solution is not unique & only exists in "one direction".

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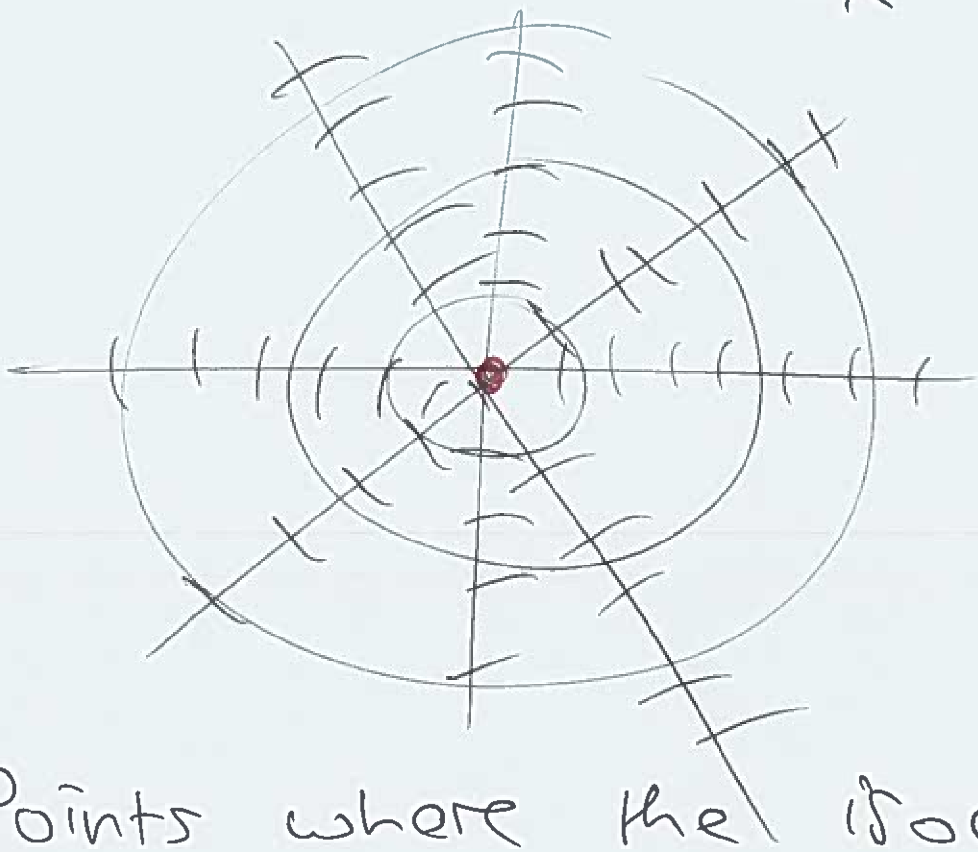
$$\frac{dy}{dx} = f(x,y) = -\frac{y}{x}$$



Points where the isoclines intersect are known as critical points.

Analytical methods for 1st order ODEs

$$\frac{dy}{dx} = f(x, y)$$

Separable ODEs

1st order ODE is separable if it can be transformed into

$$g(y)\frac{dy}{dx} = h(x)$$

$$\int g(y(x))\frac{dy}{dx}\,dx = \int h(x)\,dx$$

change of variables

$$\int g(y)\,dy = \int h(x)\,dx + A$$

Note: may not be able to solve this for $y(x)$ explicitly

Example: $\frac{dy}{dx} = -\frac{x}{y}$

revisited $y(x=1) = 0$

$$y\frac{dy}{dx} = -x$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} + \frac{x^2}{2} = \underbrace{A}_{\frac{1}{2}R^2} > 0$$

$$x^2 + y^2 = R^2$$

$$y(x) = \pm\sqrt{R^2 - x^2}$$

non-uniqueness!

IC: $y(x=1) = 0$

$$y(x) = \pm\sqrt{1-x^2}$$

Example:

$$\frac{dy}{dx} = \underbrace{e^{x-y}}_{e^{x}e^{-y}} \qquad y(x_0) = y_0$$

$$e^{y}\frac{dy}{dx} = e^{x}$$

$$\int e^{y}\,dy = \int e^{x}\,dx$$

$$e^{y} = e^{x} + A$$

$$y(x) = \ln\left(e^{x} + A\right)$$

Now apply IC (messy)

Alternative: Definite integral

$$\int_{y_0}^{y(x)} e^{\eta}\,d\eta = \int_{x_0}^{x} e^{\xi}\,d\xi$$

$$e^{y(x)} - e^{y_0} = e^{x} - e^{x_0}$$

$$e^{y(x)} = e^{x} - e^{x_0} + e^{y_0}$$

$$\underline{\underline{y(x) = \ln\left(e^{x} - e^{x_0} + e^{y_0}\right)}}$$

Example:

$$(1-y^2)\sin x \frac{dy}{dx} - y\cos x = 0 \qquad \Big| \cdot \frac{1}{y\sin x}$$

$$\frac{1-y^2}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot(x)$$

$$\int\left(\frac{1}{y} - y\right)dy = \int \cot(x)\,dx + A$$

$$\ln|y| - \frac{1}{2}y^2 = \ln|\sin x| + A$$

cannot be solved explicitly

for $y(x)$ but

$$x = \arcsin\left(B|y|e^{-\frac{y^2}{2}} \right)$$

EXERCISE

1st order ODEs of homogeneous type

ODE is of homog. type if it can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Trick:

$$\frac{y(x)}{x} = z(x)$$

$$y(x) = x\, z(x)$$

$$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\underbrace{z + x\frac{dz}{dx} = f(z)}$$

Separable ODE for $z(x)$

$$\frac{dz}{dx} = \frac{f(z) - z}{x}$$

$$\int \frac{dz}{f(z) - z} = \int \frac{dx}{x} + A$$

DON'T MEMORISE THIS!

EXAMPLE:

$$x^2 \frac{dy}{dx} + y^2 - xy = 0 \quad \Big| \cdot \frac{1}{x^2}$$

Note: $x \neq 0$

Note: $y \equiv 0$ is soln.

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$$\frac{dy}{dx} + \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right) = 0$$

$$z(x) = \frac{y(x)}{x} \; ; \; y(x) = z(x)\, x$$

$$\frac{dy}{dx} = x \frac{dz}{dx} + z$$

$$\underbrace{x \frac{dz}{dx} + \cancel{z}}_{\frac{dy}{dx}} + z^2 - \cancel{z} = 0$$

$$\int -\frac{1}{z^2}\, dz = \int \frac{1}{x}\, dx$$

$$\frac{1}{z} = \boxed{\ln|x| + C' = \frac{x}{y}}$$

$$y(x) = \frac{x}{\ln|x| + C'} \quad (*)$$

Note: The soln $y \equiv 0$ is not a special case of (*). This is a feature of nonlin. eqns.

1st order linear ODEs

$$a(x)\frac{dy}{dx} + b(x)y = c(x)$$

Rewrite in standard form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Can be solved by the "integrating factor"

Motivation:

Multiply ODE by $\mu(x)$ & integrate

$$\underbrace{\int \left(y'\mu + y \underbrace{p\mu}_{\mu'} \right) dx}_{\int \frac{d}{dx}(y\mu)\, dx} = \int \mu q \, dx$$

$$y\mu$$

To make this work

choose $\mu(x)$ s. th:

$$\frac{d\mu}{dx} = p(x)\mu(x)$$

$$\int \frac{1}{\mu} d\mu = \int p\, dx$$

$$\ln \mu = \int p\, dx$$

$$\mu(x) = \exp\left(\int p(x)\, dx \right)$$

$\hookrightarrow$ given!

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<u>Note:</u> we can drop the constant of integration here.

<u>Procedure:</u>

(1) Determine integrating factor

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

(2) Multiply ODE by $\mu(x)$ & integrate:

$$\int \underbrace{(y'\mu + y p \mu)}_{\frac{d}{dx}(\mu y)}\,dx = \int \mu q\,dx$$

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x) q(x)\,dx + C\right)$$

Example:

$$\underbrace{y' - xy}_{} = \underbrace{x}_{}$$

$$p(x) = -x \qquad q(x) = x$$

① $\mu(x) = \exp\left(\int p(x)\,dx\right)$

$$= \exp\left(\int -x\,dx\right)$$

$$\mu(x) = \exp\left(-\frac{1}{2}x^2\right)$$

②

$$\int\left(y'\exp\left(-\frac{1}{2}x^2\right) - xy\exp\left(-\frac{1}{2}x^2\right)\right)dx$$

$$\int \frac{d}{dx}\left(y(x)\exp\left(-\frac{1}{2}x^2\right)\right)dx$$

$$y(x)\exp\left(-\frac{1}{2}x^2\right) = \int q(x)\mu(x)\,dx$$

$$= \int x\exp\left(-\frac{1}{2}x^2\right)dx$$

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$$y(x) \exp\left(-\frac{1}{2}x^2\right) = -\exp\left(-\frac{x^2}{2}\right) + C$$

$$y(x) = -1 + C \exp\left(\frac{x^2}{2}\right).$$