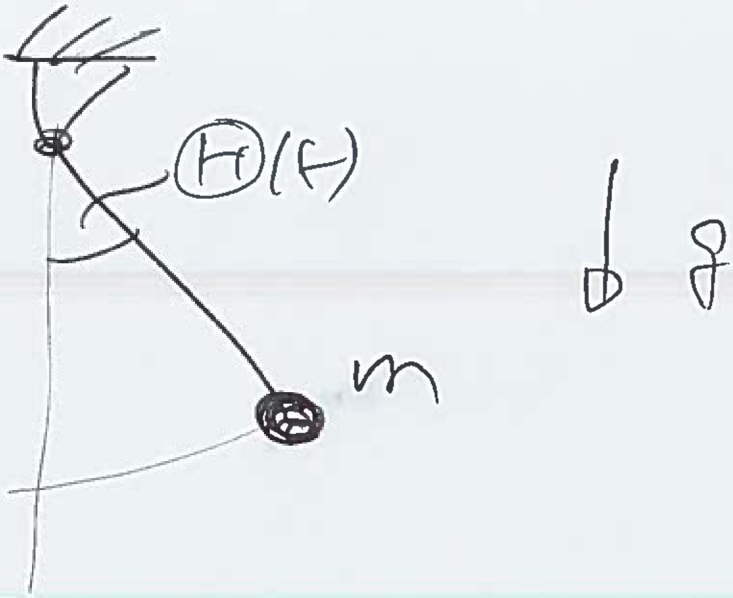



$$\boxed{\begin{aligned} &\ddot{\Theta} + \omega^2 \sin(\Theta) = 0 \\ &\underline{\text{IC}}: \ \Theta(t=0) = \varepsilon \\ &\qquad \dot{\Theta}(t=0) = 0 \end{aligned}}$$

$$\omega^2 = \frac{g}{L}$$

Non-linear IVP :
- E&U ?
- How to find solution ?

<u>E&U:</u> IVP ✓

$$\ddot{\Theta} = \underbrace{-\omega^2 \sin\Theta}$$

$$f(\cancel{t}, \Theta, \cancel{\dot{\Theta}}) = -\omega^2 \sin\Theta$$

$$\frac{\partial f}{\partial \Theta} = -\omega^2 \cos\Theta$$

$$\frac{\partial f}{\partial \dot{\Theta}} = 0$$

## Existence and Uniqueness for *non-linear* 2nd order ODEs

Consider the *non-linear* second-order ODE

$$y'' = f(x, y, y') \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants $X, Y$ and $Z$, and the function $f(x, y, y')$, are given.

### Theorem

If $f(x, y, y')$ and $\frac{\partial f(x,y,y')}{\partial y}$ and $\frac{\partial f(x,y,y')}{\partial y'}$ are continuous functions of $x, y$ and $y'$ in a region $0 < |x - X| < a$, $0 < |y - Y| < b$ and $0 < |y' - Z| < c$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

### Notes:

- The statement is easily generalised to (even) higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

are all continuous fcts of $t, \Theta, \dot{\Theta}$ (2

$\Longrightarrow$ unique soln. exists for small times $t > 0$

---

$$\ddot{\Theta} + \omega^2 \sin\Theta = 0$$

From numerical solution we note that $|\Theta(t)| \le \varepsilon$
In this regime:

$$\sin\Theta = \Theta \underbrace{- \frac{1}{3!}\Theta^3 + \frac{1}{5!}\Theta^5 - \dots}_{\text{neglect these}}$$

$$\boxed{\begin{aligned} &\ddot{\Theta} + \omega^2 \Theta = 0 \\ &\Theta(t=0) = \varepsilon \\ &\dot{\Theta}(t=0) = 0 \end{aligned}} \Longrightarrow \Theta(t) = \varepsilon \cos(\omega t)$$

Motivated by the form of the solution for $|\Theta| \ll 1$ we pose an ansatz of the form

$$\Theta(t) = \varepsilon\Theta_1(t) + \varepsilon^2\Theta_2(t) + \dots \quad (*)$$

EXERCISE: Include $\Theta_0(t)$ & find that $\Theta_0(t) = 0$.

<u>Note:</u> In nonlinear problems it is often helpful to expand nonlinear fcts of the (small) argument first.

$$\ddot{\Theta} + \omega^2\underbrace{\left(\Theta - \frac{1}{3!}\Theta^3 + \frac{1}{5!}\Theta^5 - \dots\right)}_{\sin\Theta} = 0$$

For brevity set $\omega = 1$.

$(*)$ into

$$\underbrace{\varepsilon \ddot{\Theta}_1 + \varepsilon^2 \ddot{\Theta}_2 + \varepsilon^3 \ddot{\Theta}_3 + \dots}_{\ddot{\Theta}} +$$

$$\left.\begin{aligned} &\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \varepsilon^3 \Theta_3 + \dots \\ &- \frac{1}{6}\left(\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \varepsilon^3 \Theta_3 + \dots\right)^3 \\ &+ \frac{1}{120}\left(\varepsilon \Theta_1 + \dots\right)^5 - \dots \end{aligned}\right\} \sin\Theta$$

---

Side calculation:

$$\left(\varepsilon \Theta_1 + \dots\right)^3 =$$

$$\left(\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \varepsilon^3 \Theta_3 + \dots\right)$$
$$\left(\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \varepsilon^3 \Theta_3 + \dots\right)$$
$$\left(\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \varepsilon^3 \Theta_3 + \dots\right) =$$

$$= \varepsilon^3 \Theta_1^3 + 3 \varepsilon^4 \Theta_1^2 \Theta_2 + \dots$$

5

$$\varepsilon\ddot{\Theta}_1 + \varepsilon^2\ddot{\Theta}_2 + \varepsilon^3\ddot{\Theta}_3 + \dots$$
$$+ \varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \dots$$
$$-\frac{1}{6}\left(\varepsilon^3\Theta_1^3 + 3\varepsilon^4\Theta_1^2\Theta_2 + \dots\right)$$
$$+\frac{1}{120}\left(\varepsilon^5\Theta_1^5 + \dots\right) = 0$$

IC:

$$\boxed{\Theta(t=0) = \varepsilon}$$

$$\varepsilon\Theta_1(t=0) + \varepsilon^2\Theta_2(t=0) + \varepsilon^3\Theta_3(t=0) + \dots = \varepsilon$$

$$\varepsilon\left(\Theta_1(t=0) - 1\right) +$$
$$\varepsilon^2\left(\Theta_2(t=0)\right) +$$
$$\varepsilon^3\left(\Theta_3(t=0)\right) + \dots = 0$$

$$\boxed{\dot{\Theta}(t=0) = 0}$$

$$\varepsilon\left(\dot{\Theta}_1(t=0)\right) +$$
$$\varepsilon^2\left(\dot{\Theta}_2(t=0)\right) +$$
$$\varepsilon^3\left(\dot{\Theta}_3(t=0)\right) + \dots = 0$$

Now collect powers of $\varepsilon$ in ODE:

$$\boxed{\varepsilon\left(\ddot{\Theta}_1+\Theta_1\right)+\varepsilon^2\left(\ddot{\Theta}_2+\Theta_2\right)+\varepsilon^3\left(\ddot{\Theta}_3+\Theta_3-\frac{1}{6}\Theta_1^3\right)+\dots=0}$$

$\underline{\varepsilon:}$

$$\boxed{\begin{aligned}&\ddot{\Theta}_1+\Theta_1=0\\&\Theta_1(t=0)=1\\&\dot{\Theta}_1(t=0)=0\end{aligned}}$$

$\Theta_1(t)=\cos(t)$

$\underline{\varepsilon^2:}$

$$\boxed{\begin{aligned}&\ddot{\Theta}_2+\Theta_2=0\\&\Theta_2(t=0)=0\\&\dot{\Theta}_2(t=0)=0\end{aligned}}$$

$\Theta_2(t)=0$

$\underline{\varepsilon^3:}$

$$\begin{aligned}&\ddot{\Theta}_3+\Theta_3=\frac{1}{6}\Theta_1^3\\&\Theta_3(t=0)=0\\&\dot{\Theta}_3(t=0)=0\end{aligned}$$

**Note:** Sequence of <u>linear</u> problems!

$\Theta_1(t) = \cos(t)$

$\Theta_2(t) = 0$

$$\boxed{\begin{gathered}\ddot{\Theta}_3 + \Theta_3 = \frac{1}{6}\cos^3(t) \\ \Theta_3(t=0) = 0 \\ \dot{\Theta}_3(t=0) = 0\end{gathered}}$$

$$\Theta_3(t) = \frac{1}{192}\Big(\cos(t) - \cos(3t)\Big) + \frac{1}{16}\, t\sin(t)$$

(EXERCISE)

$$\Theta(t;\varepsilon) = \varepsilon\cos(t) + \varepsilon^3\left(\frac{1}{192}\Big(\cos(t) - \cos(3t)\Big) + \frac{1}{16}\, t\sin(t)\right) + \dots$$

## [Numerical] experiment: Finite-amplitude oscillation of an undamped pendulum

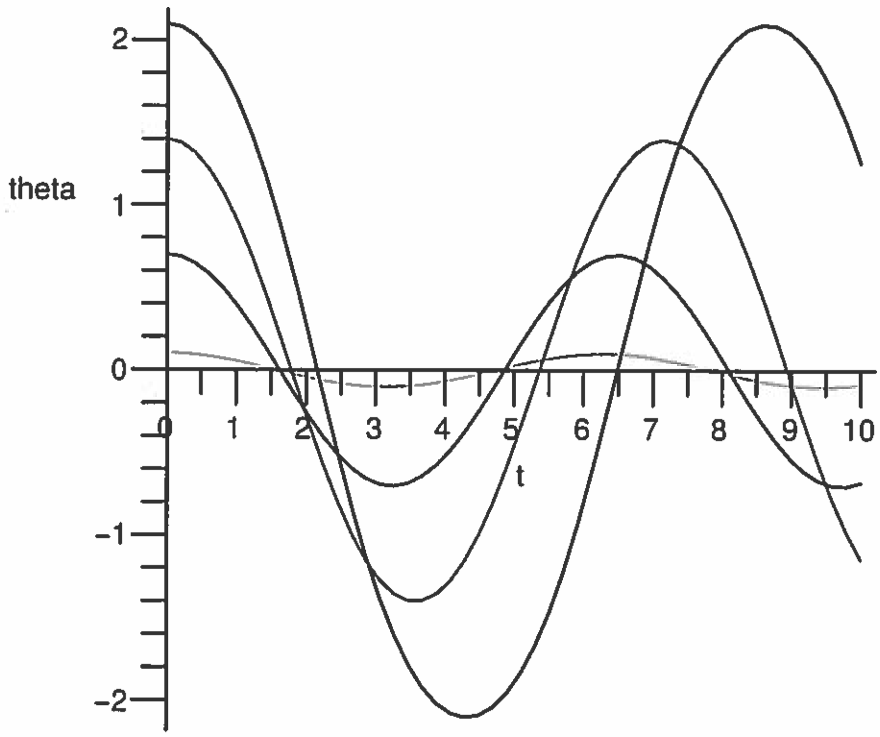
- Governing (non-linear!) ODE:

$$\ddot{\theta} + \sin\theta = 0$$

subject to the initial conditions

$$\theta(t = 0) = \epsilon \quad \text{and} \quad \dot{\theta}(t = 0) = 0.$$

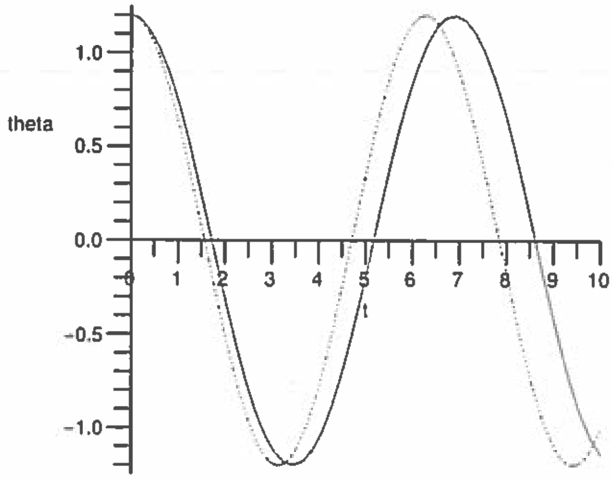
- Plot for $\epsilon = 0.1, 0.7, 1.4, 2.1$:



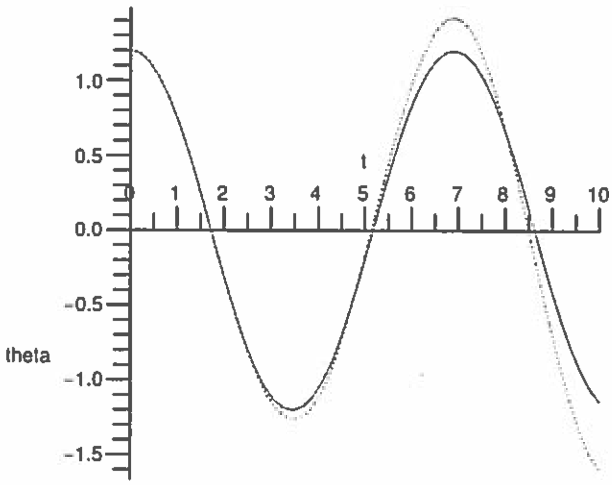

- **Observation:** Period of the oscillation increases for larger amplitudes.

## Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$

- One-term perturbation solution (red), exact solution (green):

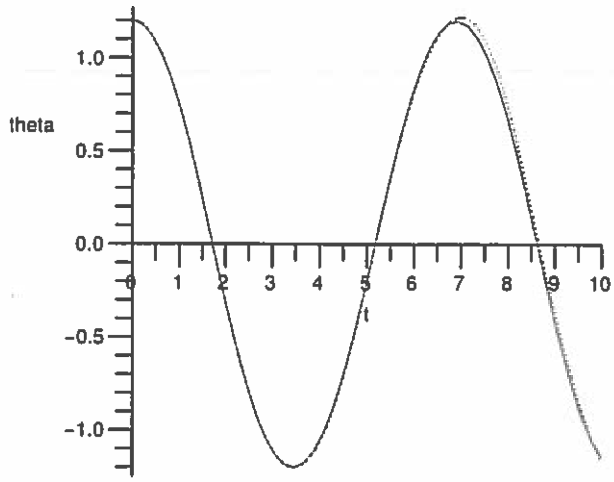



- Two-term perturbation solution (red), exact solution (green):

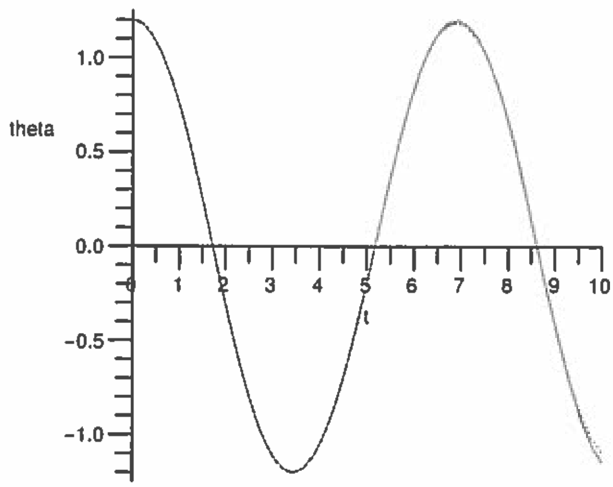


## Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$ (cont.)

- Three-term perturbation solution (red), exact solution (green):

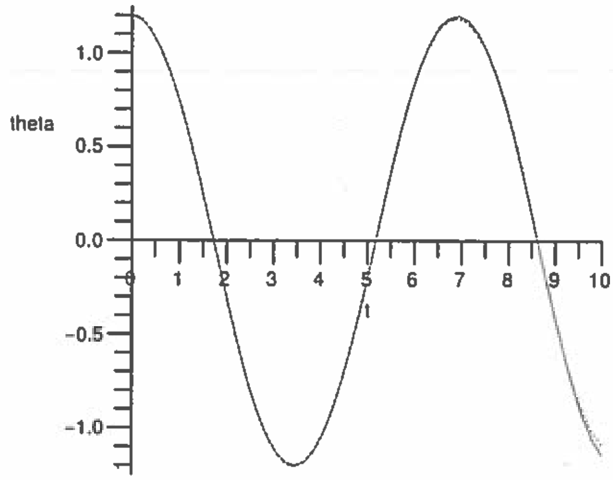



- Four-term perturbation solution (red), exact solution (green):

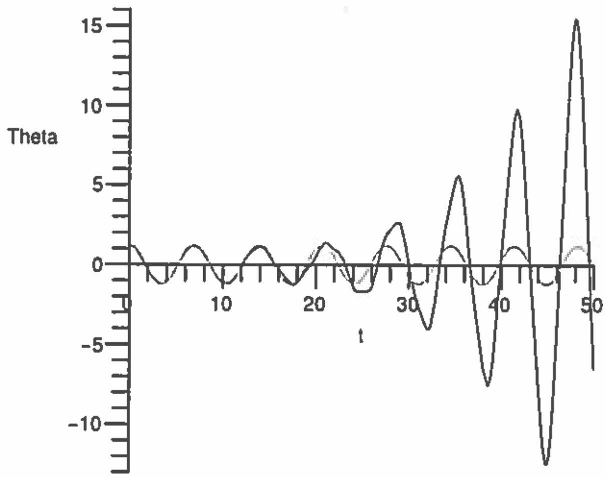


## Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$ (cont.)

- Four-term perturbation solution (red), exact solution (green):




- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:



**"Multinomial expansions"**

- One tedious task that one tends to face regularly when using perturbation methods is that of raising a power series in $\epsilon$ to some integer power

$$S = \left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^n, \tag{1}$$

  and collecting the terms multiplied by the same power of $\epsilon$, i.e. re-writing $S$ in the form

$$S = S_0(x_0) + \epsilon\, S_1(x_0, x_1) + \epsilon^2\, S_2(x_0, x_1, x_2) + \ldots \tag{2}$$

  where the functions $S_i(x_0, x_1, \ldots)$ do not depend on $\epsilon$.

- Formally, the expansion of $S$ may be obtained by using the "multinomial series" (a generalisation of the binomial series) as

$$\left(a_1 + a_2 + \ldots + a_k\right)^n = \sum_{\substack{n_1, n_2, n_3, \ldots, n_k \in \mathbb{N}_0 \\ n_1 + n_2 + \ldots + n_k = n}} \frac{n!}{n_1!\, n_2! \ldots n_k!}\, a_1^{n_1}\, a_2^{n_2} \ldots a_k^{n_k}$$

  see, e.g. `http://mathworld.wolfram.com/MultinomialSeries.html`

- However, we usually only need the first few terms in (2) for low-ish powers of $n$. Here they are:

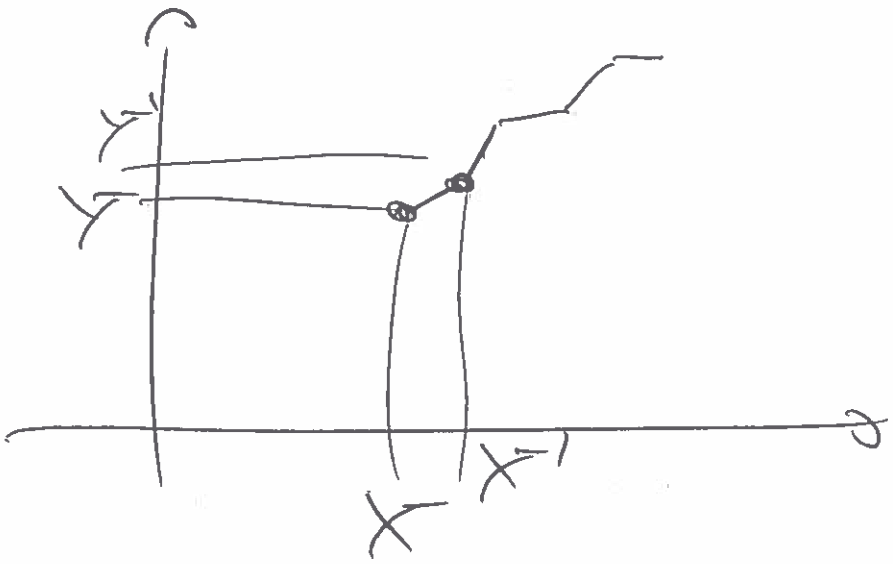
$$\begin{aligned}
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^2 &= \left(x_0^2\right) + \epsilon\left(2x_0\, x_1\right) + \epsilon^2\left(x_1^2 + 2x_0\, x_2\right) + \ldots \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^3 &= \left(x_0^3\right) + \epsilon\left(3x_0^2\, x_1\right) + \epsilon^2\left(3x_0 x_1^2 + 3x_0^2\, x_2\right) + \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^4 &= \left(x_0^4\right) + \epsilon\left(4x_0^3\, x_1\right) + \epsilon^2\left(4x_0^3 x_2 + 6x_0^2\, x_1^2\right) +
\end{aligned}$$

- **Exercise:** Convince yourself that you understand how these terms arise. **Hint:** Either use the multinomial series given above, or write $S$ explicitly as a product of $n$ power series [e.g. for $n = 2:\ \ S = (x_0 + \epsilon\, x_1 + \ldots)(x_0 + \epsilon\, x_1 + \ldots)$] and inspect which combination of terms gives rise to what powers of $\epsilon$.

- **Relax!** In an exam these expressions would be provided!

## "Bootstrapping"

The theorem only guarantees the existence and uniqueness in the "vicinity" of the initial condition. However, if you can show that the function $f(x, y, y')$ and its derivatives are "well behaved" (in the sense of the theorem), for *any* values of $x, y$ and $y'$, then the repeated application of the theorem guarantees the existence and uniqueness of the solution for *all* values of $x$.




but it's more subtle than that!

$\ddot{x} + x = -\frac{1}{2}\, t \sin(t)$

```
> restart; read("ode.map");
```

1

```
>
>
#----------------------------------------------------
# The ODE (second order correction for weakly damped
# mass spring damper system)
#----------------------------------------------------
> ode:=diff(x(t),t$2)+x(t)=-1/2*t*sin(t);
```

$$ode := \frac{d^2}{dt^2}\, x(t) + x(t) = -\frac{1}{2}\, t \sin(t)$$

```
>
#----------------------------------------------------
# ...and its homogeneous counterpart
#----------------------------------------------------
> ode_h:=diff(x(t),t$2)+x(t);
```

$$ode\_h := \frac{d^2}{dt^2}\, x(t) + x(t)$$

```
>
#----------------------------------------------------
# Solve the bloody thing using maple
#----------------------------------------------------
> dsolve(ode,x(t));
```

$$x(t) = \sin(t)\, \_C2 + \cos(t)\, \_C1 - \frac{1}{8}\, t\, (-\cos(t)\, t + \sin(t))$$

```
>
>
#----------------------------------------------------
# Now do it "by hand".
#----------------------------------------------------
>
#----------------------------------------------------
# Here's the straightforward ansatz: multiple of rhs
# is not a solution of the homogeneous ODE, but creates
# a new linearly independent function: cos(t).
#----------------------------------------------------
> x_p:=A*t*sin(t);
```

$$x\_p := A\, t \sin(t)$$

```
> eval(subs(x(t)=x_p,ode_h));
```

$$2\, A \cos(t)$$

```
> eval(subs(x(t)=x_p,ode));
```

$$2\, A \cos(t) = -\frac{1}{2}\, t \sin(t)$$

```
>
>
```

```
>
#-----------------------------------------------
# ... so we should add cos(t), but this won't work
# because it solves the homogeneous ODE, so multiply
# by t first and then add:
#-----------------------------------------------
> x_p:=A*t*sin(t)+B*t*cos(t);
```

$$x\_p := A\,t\sin(t) + B\,t\cos(t)$$

```
> eval(subs(x(t)=x_p,ode_h));
```

$$2\,A\cos(t) - 2\,B\sin(t)$$

```
> eval(subs(x(t)=x_p,ode));
```

$$2\,A\cos(t) - 2\,B\sin(t) = -\frac{1}{2}\,t\sin(t)$$

```
>
>
>
#-----------------------------------------------
# Of course, that now simply produces sin(t), which
# we ought to add to the ansatz. But it's a solution
# of the homogeneous ODE, so multiply by t and then
# add. But, hang on, that's what we started with.
# AAAARGH.
#
# Solution: Need to increase the power of t:
#-----------------------------------------------
> x_p:=A*t*sin(t)+B*t^2*cos(t);
```

$$x\_p := A\,t\sin(t) + B\,t^2\cos(t)$$

```
> eval(subs(x(t)=x_p,ode_h));
```

$$2\,A\cos(t) + 2\,B\cos(t) - 4\,B\,t\sin(t)$$

```
> eval(subs(x(t)=x_p,ode));
```

$$2\,A\cos(t) + 2\,B\cos(t) - 4\,B\,t\sin(t) = -\frac{1}{2}\,t\sin(t)$$

```
>
>
#-----------------------------------------------
# ...and this works for B=1/8 and A=-B
#-----------------------------------------------
> subs(A=-1/8,B=1/8,eval(subs(x(t)=x_p,ode)));
```

$$-\frac{1}{2}\,t\sin(t) = -\frac{1}{2}\,t\sin(t)$$

```
>
>
```