Perturbation methods

Idea:

Problem: $\mathcal{F}(x;\varepsilon) = 0$

solution $x = x(\varepsilon)$

for small ε:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

(Taylor) power series.

use as ansatz!

Into eqn.; expand in powers of ε & set largest terms (those multiplying lowest powers of ε) to zero.

$\Rightarrow$ Hierarchy of eqns for $x_0, x_1, x_2, \dots$

An ODE example

$$\boxed{\begin{aligned} &\ddot{x} + \varepsilon \dot{x} + x = 0 \\ &\text{IC:}\quad x(t=0) = 1 \\ &\qquad\ \dot{x}(t=0) = 0 \end{aligned}}\qquad f(x;\varepsilon)$$

Assume we don't know the solution but note:

If $\varepsilon = 0$:

$$\boxed{\begin{aligned} &\ddot{x} + x = 0 \\ &x(t=0) = 1 \\ &\dot{x}(t=0) = 0 \end{aligned}} \longrightarrow x(t) = \cos(t)$$

Ansatz:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$$

$$\dot{x} = \dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \dots$$

$$\ddot{x} = \ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots$$

into ODE $\ddot{x} + \varepsilon \dot{x} + x = 0$

$$\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots +$$

$$\varepsilon(\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \dots) +$$

$$x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots = 0$$

collect powers of ε:

$$\boxed{(\ddot{x}_0 + x_0) + (\ddot{x}_1 + \dot{x}_0 + x_1)\varepsilon + (\ddot{x}_2 + \dot{x}_1 + x_2)\varepsilon^2 + \dots = 0}$$

Do the same thing with the ICs:

$$x(t=0) = 1$$

$$x_0(t=0) + \varepsilon x_1(t=0) + \varepsilon^2 x_2(t=0) + \dots = 1$$

collect powers of ε:

$$\boxed{(x_0(t=0) - 1) + (x_1(t=0))\varepsilon + (x_2(t=0))\varepsilon^2 + \dots = 0}$$

$\dot{x}(t=0) = 0$:

$$\boxed{\dot{x}_0(t=0) + \dot{x}_1(t=0)\,\varepsilon + \dot{x}_2(t=0)\,\varepsilon^2 + \ldots = 0}$$

Now collect largest terms from expanded ODE & the ICs:

ε^0:

$$\boxed{\begin{aligned} &\ddot{x}_0 + x_0 = 0 \\ &x_0(t=0) = 1 \\ &\dot{x}_0(t=0) = 0 \end{aligned}}$$

This is the orig. problem for $\varepsilon = 0$.

"cos"

$$\boxed{x_0(t) = \cos(t)}$$

ε^1:

$$\boxed{\begin{aligned} &\ddot{x}_1 + x_1 = -\dot{x}_0 \\ &x_1(t=0) = 0 \\ &\dot{x}_1(t=0) = 0 \end{aligned}}$$

$\to x_1(t)$

$\underline{\varepsilon^2:}$

$$\boxed{\begin{aligned} &\ddot{x}_2 + x_2 = -\dot{x}_1 \\ &x_2(t=0) = 0 \\ &\dot{x}_2(t=0) = 0 \end{aligned}} \longrightarrow x_2(t)$$

$\vdots$

etc

So the structure is as in the algebraic example:

- "Leading-order problem" is the original problem for $\varepsilon = 0$.
- we obtain a hierarchy of IVPs for the corrections $x_1(t)$, $x_2(t)$, ...

To solve the eqns note that all the problems are linear IVPs:

$$x_i(t) = x_i^{(p)}(t) + \underbrace{x_i^{(H)}(t)}_{\text{General solution of the homogeneous ODE}}$$

(Same in all cases)

ε^0: $\ddot{x}_0 + x_0 = 0$

$x_0(\cancel{t}) = 1$

$\dot{x}_0(t=0) = 0$

$$\boxed{x_0^{(H)} = A\cos(t) + B\sin(t)}$$

$x_0^{(p)} = 0$

$$x_0(t) = x_0^{(H)}(t) = A\cos(t) + B\sin(t)$$

Apply IC: $A = 1$ $B = 0$

$$\Rightarrow x_0(t) = \cos(t)$$

$\underline{\varepsilon^1:}$

$$\boxed{\ddot{x}_1 + x_1} = -\dot{x}_0 = \sin(t)$$

$$x_1(t=0) = 0$$

$$\dot{x}_1(t=0) = 0$$

$$x_1^{(H)} = A\cos(t) + B\sin(t)$$

$x_1^{(p)}$ use method of undeterm. coefficients

$$x_1^{(p)} = C t\sin(t) + D t\cos(t)$$

$\vdots$ EXERCISE

plug in: $C = 0$

$$D = -1/2$$

$$x_1(t) = A\cos(t) + B\sin(t) - \frac{1}{2}t\cos(t)$$

Apply ICs: $\left.\begin{array}{l} x_1(t=0) = 0 \\ \dot{x}_1(t=0) = 0 \end{array}\right\} \begin{array}{l} A = 0 \\ B = 1/2 \end{array}$

(8)

$$x_1(t) = \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t).$$

$\underline{\varepsilon^2:}$ $$\ddot{x}_2 + x_2 = -\dot{x}_1$$
$$= -\frac{1}{2}t\sin(t)$$

$$x_2^{(H)} = A\cos(t) + B\sin(t)$$
$$x_2^{(P)} = -\frac{1}{8}t\sin(t) + \frac{1}{8}t^2\cos(t)$$

EXERCISE.

Here $x_2^{(P)}$ actually satisfies the ICs:
$$x_2(t=0) = 0$$
$$\dot{x}_2(t=0) = 0$$

$$x_2(t) = -\frac{1}{8}t\sin(t) + \frac{1}{8}t^2\cos(t)$$

$\vdots$

$$x_3(t) = \frac{1}{16}\sin(t) - \frac{1}{48}t(3+t^2)\cos(t)$$

$\vdots$

$$X(t;\varepsilon) = \cos(t) + \varepsilon\left(\frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t)\right) + \varepsilon^2\left(-\frac{1}{8}t\sin(t) + \frac{1}{8}t^2\cos(t)\right) + \ldots$$

observations when comparing against the ~~exact~~ solution:

1. Solution converges rapidly at fixed value of t if
   - the number of terms in the perturbation solution is increased
   - ε gets smaller
2. As $t \to \infty$ the perturbation solution becomes increasingly inaccurate.
   - terms $\sim t^n$ grow & become dominant

Mechanical oscillator with weak damping

- Governing (linear) ODE:

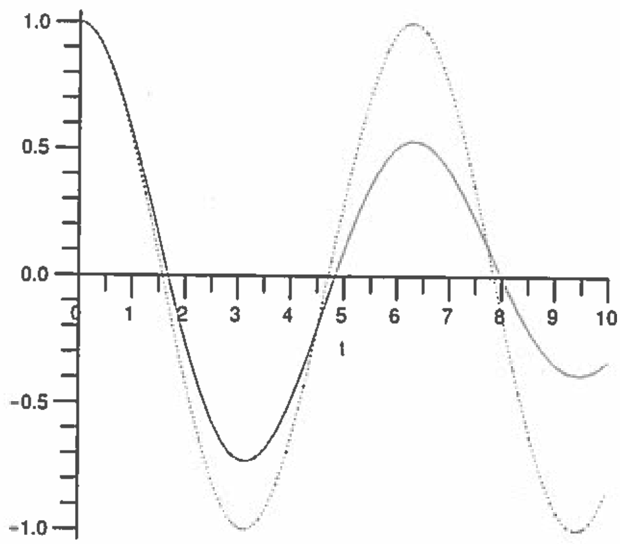
$$\ddot{x} + \epsilon\dot{x} + x = 0$$

subject to the initial conditions

$$x(t=0) = 1 \quad \text{and} \quad \dot{x}(t=0) = 0.$$

Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- One-term perturbation solution (red), exact solution (green):



Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

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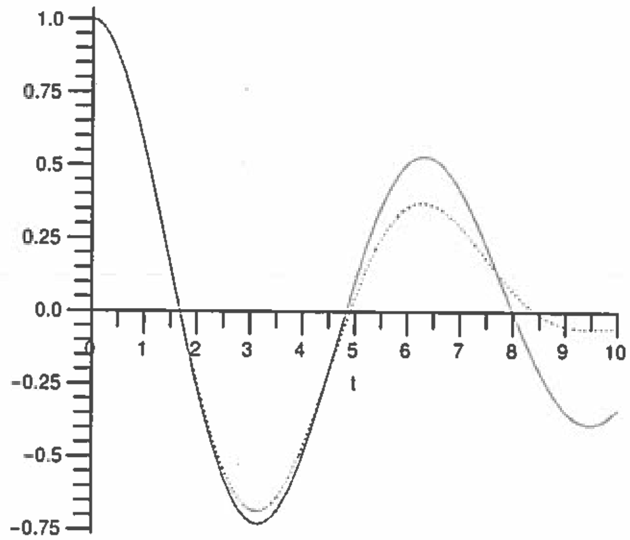
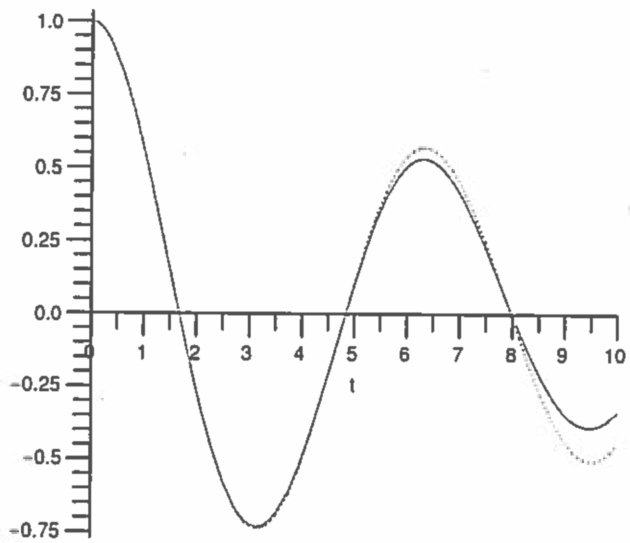
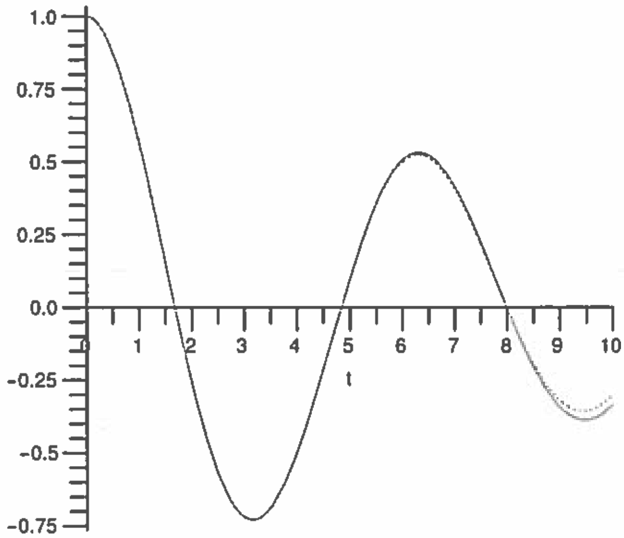


- Three-term perturbation solution (red), exact solution (green):

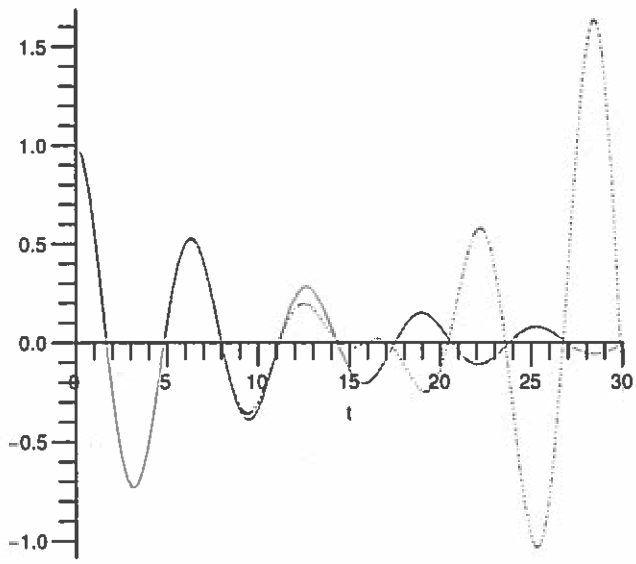


Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Four-term perturbation solution (red), exact solution (green):



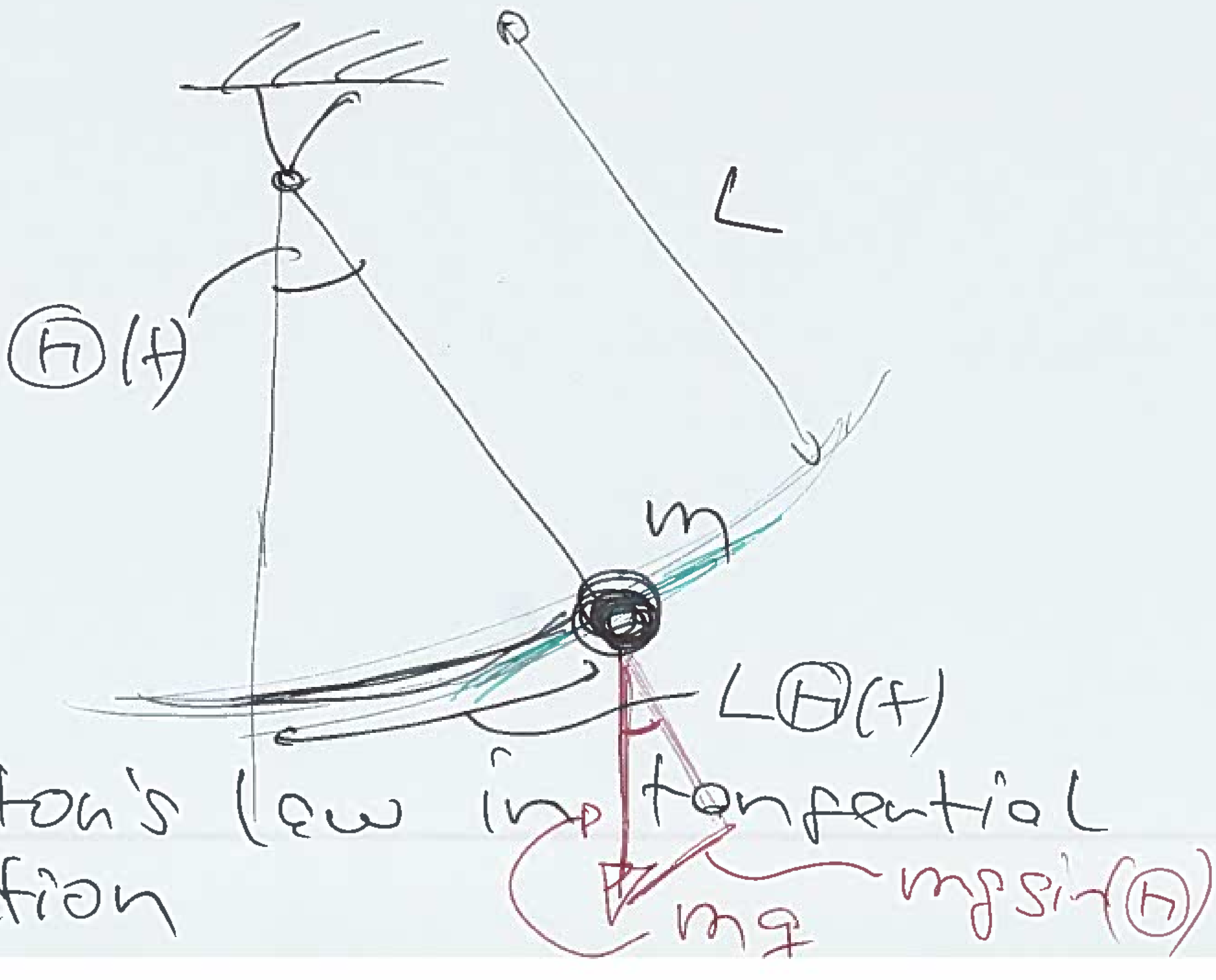
- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:



- Underlying reason: The errors commited by only satisfying the ODE up to some small but finite error accumulate.

A nonlinear example

Pendulum:



Newton's law in tangential direction

$$\cancel{m} L \frac{d^2\Theta}{dt^2} = -\cancel{m} g \sin(\Theta)$$

$$\ddot{\Theta} + \omega^2 \sin\Theta = 0$$

where $\omega^2 = \frac{g}{L}$

IC: $\Theta(t=0) = \varepsilon$

$\dot{\Theta}(t=0) = 0$