Example:

$$\ddot{y} + 3\dot{y} + y = 4 + 2t^2$$

$$= A_1 r_1(t) + A_2 r_2(t)$$

$$A_1 = 4 \quad r_1(t) = 1$$

$$A_2 = 2 \quad r_2(t) = t^2$$

Ansatz:

$$y_p = c_1 \cdot \underbrace{1}_{r_1(t)} + c_2 \underbrace{t^2}_{r_2(t)} + c_3 t$$

$$\dot{y}_p = 2c_2 t + c_3$$

$$\ddot{y}_p = 2c_2$$

into ODE:

$$\underbrace{2c_2}_{\ddot{y}} + 3\underbrace{(2c_2 t + c_3)}_{\dot{y}} + \underbrace{c_1 + c_2 t^2 + c_3 t}_{y}$$

$$\overset{!}{=} 4 + 2t^2 \qquad \forall t$$

collect lin. indep. fcts. of t: powers of t:

$$\underbrace{(2c_2' + c_1' - 4 + 3c_3')}_{=0} t^0 + \underbrace{(6c_2' + c_3')}_{=0} t^1 + \underbrace{(c_2' - 2)}_{=0} t^2 \stackrel{!}{=} 0 \quad \forall t$$

$c_2' = 2$

$6c_2' = 0$

$c_2' = 0$

$\lightning$

Does not work because differentiation of t^2 produces a term $\sim t$

$\Rightarrow$ Remedy: include this term!

with $c_3 t$ included:

$$c_2 - 2 = 0 \Rightarrow c_2 = 2$$

$$6c_2 + c_3 = 0 \Rightarrow c_3 = -12$$

$$2c_2 + c_1 - 4 + 3c_3 = 0 \Rightarrow c_1 = 36$$

So

$$\underline{\underline{y_p = 36 - 12t + 2t^2}}$$

Example:

$$\ddot{y} + 3\dot{y} = 1 + 9t^2$$

Ansatz:

$$\hat{y}_p = c_1 + c_2 t + c_3 t^2$$

$$\dot{\hat{y}}_p = + c_2 + 2c_3 t$$

$$\ddot{\hat{y}}_p = + 2c_3$$

into ODE

$$\underbrace{2c_3}_{\ddot{y}} + \underbrace{3c_2 + 6c_3 t}_{3\dot{y}} \overset{!}{=} 1 + 9t^2 \quad \forall t$$

$$\underbrace{(2c_3 + 3c_2 - 1)}_{=0} + \underbrace{(6c_3)}_{=0} t - \underbrace{9t^2}_{=0} = 0 \quad \forall t$$

Doesn't work because the constant is a soln. of the homog. ODE

Remedy: multiply ansatz by t.

$$y_p = c_1 t + c_2 t^2 + c_3 t^3$$

$$\dot{y}_p = c_1 + 2c_2 t + 3c_3 t^2$$

$$\ddot{y}_p = 0 + 2c_2 + 6c_3 t$$

into ODE

$$2c_2 + 6c_3 t + 3\left(c_1 + 2c_2 t + 3c_3 t^2\right)$$

$$\overset{!}{=} 1 + 9t^2 \qquad \forall t$$

collect:

$$\underbrace{\left(2c_2 + 3c_1 - 1\right)}_{=0} + \underbrace{\left(6c_3 + 6c_2\right)}_{=0} t + \underbrace{\left(9c_3 - 9\right)}_{=0} t^2 = 0 \qquad \forall t$$

$\Rightarrow$ $c_3 = 1$

$\Rightarrow$ $c_2 = -1$

$\Rightarrow$ $c_1 = 1$

$$\underline{\underline{y_p = t - t^2 + t^3}}$$

Sadly the method only works for a small set of RHS fct.

E.g. it does not work for

$$r(x) = \log x$$
$$r'(x) = x^{-1}$$
$$r''(x) = -x^{-2}$$
$$\vdots$$

Method only works for

$$r(x) = \underbrace{P(x)}_{\text{polynomial}} \; e^{mx} \begin{Bmatrix} \sin(nx) \\ \cos(nx) \end{Bmatrix}$$

Other method:

- Method of variation of parameters
- Power series expansions
- ...

Nonlinear ODEs

2 special cases:

1 ODEs that do not depend on y

$$y'' = f(x, \cancel{y}, y')$$

$$y'' = f(x, y')$$

Interpret as 1^{st} order ODE for $u(x) = y'(x)$

$$u' = f(x, u)$$

- solve for $u(x)$ (introduces one constant)
- solve $y'(x) = u(x)$ for $y(x)$; introduces 2^{nd} constant.

Example:

$$3(y')^2 y'' = 1$$

Subst: $y' = u$

$$3u^2 u' = 1$$

$$3u^2 \frac{du}{dx} = 1$$

Sep:

$$\int 3u^2 \, du = \int dx$$

$$u^3 = x + C$$

$$u = \boxed{(x+C)^{1/3} = \frac{dy}{dx}}$$

integrate:

$$y(x) = \frac{3}{4}(x+C)^{4/3} + D$$

② Autonomous ODEs

$$y'' = f(\cancel{x}, y, y')$$

$$y'' = f(y, y')$$

Can be reduced to 1st order ODE for $y'(x) = u(x)$ by treating $u = y'$ as a function of y rather than x.

$$y'' = \frac{du}{dx} = \frac{du}{dy}\underbrace{\frac{dy}{dx}}_{u}$$

into ODE:

$$u\frac{du}{dy} = f(y, u)$$

a 1st order ODE for $u(y)$

solve for u & then solve

$$\frac{dy}{dx} = u(y(x))$$

Example:

$$y y'' - 2(y')^2 + 2y' = 0$$

Note: $y = 0$ is a soln.

Set: $u = y'$

$$y'' = u \frac{du}{dy}$$

into ODE

$$y u \frac{du}{dy} - 2u^2 + 2u = 0$$

Note: $u = 0$ is also a soln.
so $y = C$

$$u\underbrace{\left(y \frac{du}{dy} - 2u + 2\right)}_{0} = 0$$

$$y\frac{du}{dy} = 2u - 2$$

Separate:

$$\int \frac{1}{u-1}\,du = \int \frac{2}{y}\,dy$$

$$\ln|u-1| = 2\ln|y| + \tilde{C}$$

$$= \ln y^2 + \ln D$$

$$= \ln(Dy^2)$$

$$u(y) = \boxed{1 + Dy^2 = \frac{dy}{dx}}$$

Sep. again:

$$\int dx = \int \frac{1}{1+Dy^2}\,dy$$

$$x + \hat{\hat{C}} = \frac{1}{\sqrt{D}}\arctan(\sqrt{D}\,y)$$