Constant coeffn. ODEs

$$y'' + p y' + q y = r(x)$$

$$y(x) = y_p + \underbrace{A y_1(x) + B y_2(x)}_{\checkmark}$$

y_p: For $r(x) = A e^{ax}$ (A, a given)

Try $y_p(x) = C e^{ax}$

into ODE:

$$C = \frac{A}{a^2 + pa + q}$$

What can go wrong?

What if $a^2 + pa + q = 0$,

i.e. if a is a root of the char. poly.

Try: $y = C x e^{ax}$

$y' = C e^{ax}(1+ax)$

$y'' = C e^{ax}(a(2+ax))$

into ODE:

$$y'' + py' + qy = Ae^{ax}$$

$$C\cancel{e^{ax}}\Big(\underbrace{a(2+ax)}_{y''} + p\underbrace{(1+ax)}_{y'} + q\underbrace{x}_{y}\Big) \overset{!}{=} A\cancel{e^{ax}}$$

collect x-terms

$$C\Big(x\cancel{(a^2+pa+q)} + (2a+p)\Big) \overset{!}{=} A$$

$$C = \frac{A}{2a+p}$$

$$\Rightarrow \quad y_p = \frac{A}{2a+p} x e^{ax}$$

But what if $2a+p=0$ as well?

Check this case in more detail:

- we know that a is root of the char. poly.

$$a = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

- Now we also have:

$$2a+p=0 \Rightarrow a = -\frac{p}{2}$$

$\Rightarrow$ This latest problem only arises if a is a repeated root of the char. poly.

In that case try

$$y = d\,x^2 e^{ax}$$

$$y' = d\,e^{ax}\left(2x+ax^2\right)$$

$$y'' = d\,e^{ax}\left(2+4ax+a^2x^2\right)$$

into ODE

$$C\cancel{e^{ax}}\left(\underbrace{2+4ax+a^2x^2}_{y''}+p(\underbrace{2x+ax^2}_{y'})+\underbrace{qx^2}_{y}\right) \overset{!}{=} A\cancel{e^{ax}}$$

collect powers of x:

$$C\left(x^2\cancel{(a^2+pa+q)}+x\cancel{(4a+2p)}+2\right)\overset{!}{=}A$$

In that case

$$C=\frac{A}{2}$$

$$\underline{\underline{y_p=\frac{A}{2}x^2e^{ax}}}$$

So general procedure:

$$y'' + py' + qy = Ae^{ax}$$

① Solve char. poly.

$$\lambda^2 + p\lambda + q = 0 \Rightarrow \lambda_{12}$$

& homog. soln. y_1 & y_2

② • If $a \neq \lambda_1$ & $a \neq \lambda_2$

$$y_p = Ce^{ax}$$

• If $a = \lambda_1$ or $a = \lambda_2$
($\& \ \lambda_1 \neq \lambda_2$)

$$y_p = Cxe^{ax}$$

• If $a = \lambda_1 = \lambda_2$

$$y_p = Cx^2e^{ax}$$

Note:

Have classified the ~~three~~ three cases based on the roots of the char. poly.

Reformulate: Special cases arise if RHS $r(x)$ is a solution of the homog. ODE.

Generalisation:

$$y'' + py' + qy = A_1 r_1(x) + A_2 r_2(x) + \dots + A_n r_n(x)$$

where A_i, $r_i(x)$ are given. Assume the $r_i(x)$ are lin. indep.

Fact: If $y_1(x)$ is a solution of

$$y'' + py' + qy = r_1(x)$$

and $y_2(x)$ is a solution of

$$y'' + p y' + q y = r_2(x)$$

then $A_1 y_1 + A_2 y_2$ is a soln. of

$$y'' + p y' + q y = A_1 r_1(x) + A_2 r_2(x)$$

(EXERCISE)

This suggests the ansatz:

$$y_p = \underbrace{C_1 r_1(x) + C_2 r_2(x) + \dots + C_n r_n(x)}_{\text{undet. coeffns.}}$$

<u>Plan:</u> insert into ODE, collect linearly indep. fcts & set their coefficients to zero $\Rightarrow$ n eqns for the n unknowns $C_1, \dots, C_n$

<u>Modification #1:</u> If the differentiation of any of the $r_i(x)$ produces new linearly indep. fcts add them!

$$\hat{y}_p = C_1 r_1(x) + \ldots + C_n r_n(x) + D_1 r_1'(x) + \ldots + D_n r_n'(x) + E_1 r_1''(x) + \ldots + E_n r_n''(x)$$

Note: only add new lin. indep. fcts.

what can go wrong?

If any of the $r_i(x)$ are solns. of the homogeneous ODE we have a problem.

To demonstrate consider

$$y'' + p y' + q y = A r$$

Ansatz: $y_p = C r$

into ODE:

$$C \underbrace{(r'' + p r' + q r)}_{} \overset{!}{=} A r$$

ODE cannot be solved for C if $r'' + p r' + q r = 0$

In that case choose:

$$y_p = C x r$$
$$y_p' = C(r + x r')$$
$$y_p'' = C(2r' + x r'')$$

into ODE:

$$y'' + p y' + q y = A r$$

$$C\Big(\underbrace{2r' + x r''}_{y''} + \underbrace{pr + x p r'}_{p y'}$$
$$+ \underbrace{q x r}_{q y}\Big) \overset{!}{=} A r$$

collect powers of x:

$$C\Big(x\,\cancel{(r'' + p r' + q r)} + (2r' + p r)\Big)$$
$$\overset{!}{=} A r$$

This won't work either if in addition $2r' + pr = 0$, or alternatively if r & xr are both solutions of the homog. ODE.

In that case try

$$y_p = C x^2 r$$

Stick into ODE ...

tedious algebra happens

$$C = \frac{1}{2} A$$

$$y_p = \frac{1}{2} A x^2 r$$

Modification #2:

If one of the terms in $\hat{y}_p$ is a soln. of the homog. ODE multiply that term by x^m where m is the smallest positive integer for which the product is <u>not</u> a soln. of the homog. ODE.

Examples:

$$\ddot{y} + 4y = \underbrace{\cos(3t)}_{r_1(t)} + 2\underbrace{\sin(t)}_{r_2(t)}$$

$$y_H = \hat{A}\sin(2t) + \hat{B}\cos(2t)$$

(EXERCISE)

$A_1 = 1 \qquad r_1(t) = \cos(3t)$

$A_2 = 2 \qquad r_2(t) = \sin(t)$

or

$A_2 = 12 \qquad r_2(t) = \frac{1}{6}\sin(t)$

Ansatz:

$$y_p = A\cos(3t) + B\sin(t)$$

undetermined coeffns.

$$\dot{y}_p = -3A\sin(3t) + B\cos(t)$$

$$\ddot{y}_p = -9A\cos(3t) - B\sin(t)$$

into ODE: $\ddot{y} + 4y = \cos(3t) + 2\sin(t)$

$$\underbrace{-9A\cos(3t) - B\sin(t)}_{\ddot{y}_p} \underbrace{+4A\cos(3t) + 4B\sin(t)}_{4y_p}$$

$$\overset{!}{=} \cos(3t) + 2\sin(t) \quad \forall t$$

collect linearly indep. fcts:

$$\left.\begin{aligned}\cos(3t)\underbrace{(-9A+4A-1)}_{\overset{!}{=}0} +\\ \sin(t)\underbrace{(-B+4B-2)}_{\overset{!}{=}0}\end{aligned}\right\} = 0 \quad \forall t$$

$$\Rightarrow A = -\frac{1}{5} \qquad B = \frac{2}{3}$$

Example:

$$\ddot{y} + 3\dot{y} + y = 4 + 2\underbrace{t^2}_{r_2(t)}$$

$$A_1 = 4 \qquad r_1(t) = 1$$

$$A_2 = 2 \qquad r_2(t) = t^2$$

Ansatz:

$$y_p = C_1 \cdot \underbrace{1}_{r_1(t)} + C_2 \underbrace{t^2}_{r_2(t)}$$

$$\dot{y}_p = \qquad + 2C_2 t$$

$$\ddot{y}_p = \qquad 2C_2$$

into ODE

TBC on Thu