MATH10222 Lecture Notes

This set of notes summarises the main results of the first half of the lecture MATH10222 (Calculus and Applications). Please email any corrections (yes, there might be the odd typo...) or suggestions for improvement to *M.Heil@maths.manchester.ac.uk* or see me after the lecture.

Generally, the notes will be handed out after the material has been covered in the lecture. You can also download them from the WWW:

https://personalpages.manchester.ac.uk/staff/matthias.heil/Lectures/FirstYearODEs/index.html.

This WWW page will also contain announcements, example sheets, solutions, etc.

This course does not follow any particular textbook – your lecture notes and the handouts will be completely sufficient. If you bought Stewart's textbook for the first-semester courses, you can, of course, consult it on any of the topics covered in this lecture.

If you want a concise overview of the theory plus lots and lots of worked examples, have a look at Richard Bronson's "Differential Equations" in the Schaum's Outline Series.

1 Generalities

1.1 Ordinary derivatives

- A differentiable function $y(x)$ of one independent variable, or "argument" x, has the derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) \quad \text{or} \quad y'(x)$$

(either notation meaning the same thing) which represents the rate at which $y(x)$ changes as x changes, at the point x. That is, $y'(x)$ is a function defined by the limit

$$y'(x) = \lim_{|h|\to 0} \frac{y(x+h) - y(x)}{h}.$$

- The function $y(x)$ is said to be differentiable in an interval $I \subseteq \mathbb{R}$ if this limit exists at all values of x in the interval.

- If $y'(x)$ is also differentiable, then a second derivative, $y''(x)$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x)$, can be defined by replacing y with y' in the definition. Similarly, provided it exists, a derivative of *order* n

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}(x) \quad \text{or} \quad y^{(n)}(x)$$

can be defined by repeating the differentiation n times.

- **Alternative notations:** A superscript dot or a capital D are also sometimes used to denote differentiation. A subscript (normally signifying partial differentiation) can also be used to denote ordinary differentiation. Thus derivatives of $u(t)$ can be represented by[1]

$$\dot{u}(t) = \mathrm{D}\,u(t) = u_t(t) = u^{(1)}(t) = \frac{\mathrm{d}u}{\mathrm{d}t}(t) = u'(t), \qquad \ddot{u} = \mathrm{D}^2 u = u_{tt} = u^{(2)} = \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = u''.$$

- Note that it is not necessary to write out the argument, in this case '(t)', after each of the derivatives, if it is clearly understood that u, u', u'', etc. are all functions of t.

1.2 Ordinary Differential Equations

- **Ordinary Differential Equations:** An ordinary differential equation, or ODE, relates a function $y(x)$ to x and some of its derivatives, $y'(x)$, $y''(x)$, …, $y^{(n)}(x)$. In general it has the form

$$F\big(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}\big) = 0$$

although we will often assume that it can be written in the form

$$y^{(n)} = f\big(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)}\big).$$

Much of this course will be devoted to studying ordinary differential equations of this type. The *order* of an ordinary differential equation is the order of the highest derivative appearing in the equation.

- **Solutions:** A solution of the ODE $F\big(x, y, y^{(1)}, \ldots, y^{(n)}\big) = 0$ on an interval I is any function $\phi(x)$ such that ϕ and all of the derivatives $\phi^{(1)}$, $\phi^{(2)}$, …, $\phi^{(n)}$ exist on I and

$$F\big(x, \phi, \phi^{(1)}, \ldots, \phi^{(n)}\big) = 0 \quad \text{for all} \quad x \in I.$$

- **Linear ODEs:** Linear ODEs have a rich theoretical foundation and they are simpler to analyse than nonlinear ODEs. The ordinary differential equation for $y(x)$

$$F\big(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}\big) = 0$$

[1]Yet other notations for derivatives are sometimes encountered in various texts.

is linear if $F\big(x, y, y^{(1)}, \ldots, y^{(n)}\big)$ is linear in y and all derivatives of y (namely, all of the arguments y, $y^{(1)}$, $y^{(2)}$, …, $y^{(n)}$). In other words, it is linear if the ODE can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y - g(x) = 0$$

in which all of the coefficients g, a_0, a_1, …, a_n depend only on x (that is, they do not depend on y or any derivatives of y).

- **Autonomous ODEs:** An ODE for $y(x)$ of the form $F\big(y, y^{(1)}, \ldots, y^{(n)}\big) = 0$ in which the independent variable x does not appear, is said to be autonomous.
- **Examples:** The ODE $(1-t)u'' - tu = 0$ for $u(t)$ is non-autonomous and linear; the ODE $v'(z) + v'^2(z) - v(z) = 0$ for $v(z)$ is autonomous and nonlinear.

1.3 Some basic preliminaries

We will now look at a number of simple examples and basic features of ordinary differential equations, the use of additional information at particular points, and the way in which solutions of ODEs depend on such data.

1.3.1 Existence of solutions

- It is not always obvious that solutions will exist at all. For example, the following ODE for $y(x)$

$$y' + 1/y' = 0$$

cannot have a real-valued solution. The value of the derivative of any solution would have to be either $y' = i$ or $y' = -i$, where $i = \sqrt{-1}$, which is not real. Therefore, if we are dealing only with real functions, then there is no solution.

1.3.2 Non-uniqueness

- If we solve the very simple ODE $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, for $y(x)$, by integrating successively, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = A_1 \quad \text{and} \quad y = A_1x + A_2$$

where A_1 and A_2 are "arbitrary" constants of integration. That is, the function $A_1x + A_2$ provides a solution of $y'' = 0$ whatever constant values are chosen for A_1 and A_2.

In fact, every solution of this ODE has the form $A_1x + A_2$ for constant values of A_1 and A_2. The solution is not unique since a different choice of values for A_1 and A_2 provides a different solution.

1.3.3 Boundary and initial conditions

- A unique solution can only therefore arise if there are additional constraints on the allowed values of the solution. In general, for an n^{th} order ODE, there must be n independent constraints, if there is to be a unique solution. These constraints are usually provided by "boundary conditions" or "initial conditions".
- **Example 1:** As has been seen, the simple ordinary differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ has the solution $y = A_1x + A_2$ for arbitrary values of A_1 and A_2. If we impose the two constraints

$$y(0) = 1 \quad \text{and} \quad y(1) = 0$$

then we find that:

$$\left.\begin{array}{ll} y(0) = 1: & A_1 \times 0 + A_2 = 1 \\ y(1) = 0: & A_1 \times 1 + A_2 = 0 \end{array}\right\} \quad \Longrightarrow \quad A_1 = -1 \quad \text{and} \quad A_2 = 1.$$

There is, therefore, only one solution, namely $y = -x + 1$, that satisfies the two constraints $y(0) = 1$ and $y(1) = 0$.

- **Example 2:** The ordinary differential equation $xy'' - (1+x)y' + y = 0$ has the solution $y = B_1\,\mathrm{e}^x + B_2(1+x)$ for arbitrary values of B_1 and B_2. If we impose the constraints

$$y(1) = -1 \quad \text{and} \quad y'(1) = 0$$

then we find that:

$$\left.\begin{array}{lll} y(1) = -1 : & B_1\,\mathrm{e} + 2B_2 = -1 \\ y'(1) = 0 \;\; : & B_1\,\mathrm{e} + \;B_2 \;= 0 \end{array}\right\} \quad \Longrightarrow \quad B_1 = \mathrm{e}^{-1} \quad \text{and} \quad B_2 = -1.$$

There is, therefore, only one solution, namely $y = \mathrm{e}^{x-1} - (1+x)$, that satisfies the two constraints $y(1) = -1$ and $y'(1) = 0$.

- **Initial value problem (IVP):** When all of the constraints are specified at the same value of x, the problem is called an initial value problem, as in *Example 2* above. In applications, initial value problems typically represent evolutionary problems in which the initial conditions specify the initial state of a system while the ODE describes its rate of change.

- **Boundary value problem (BVP):** When constraints are specified at two, or more, different values of x, for example at each end of an interval I, then the problem is called a boundary value problem, as in *Example 1* above. In applications, boundary value problems typically represent spatial problems in which the boundary conditions specify the state of a system at its boundary while the ODE describes its behaviour in the interior.

- **Note:** A first-order ordinary differential equation with one constraint is, automatically, an initial value problem.

- **Exceptions:** In these examples, we have seen that n independent constraints lead to a unique solution of an n^{th} order ordinary differential equation. However, this is not always the case!

1.3.4 Basic questions

- Given an initial value problem or a boundary value problem we would like to know the answers to the following questions:

  EXISTENCE: Is there any solution at all?
  An ODE arising from a physical problem should have at least one solution if the mathematical form of the model is correct.

  UNIQUENESS: How many solutions are there, or how many constraints are needed to obtain a unique solution?

  PROPERTIES: What are the properties of the solutions?
  Perhaps even without finding any solutions can we determine their general behaviour?
  How might different solutions be related to each other?

  SOLUTION: How can we find the solutions?
  (analytical methods, numerical techniques, power-series expansions, etc.)

- The final two questions are the main practical topic to be pursued in the rest of this course. The first two questions are partly answered by the existence and uniqueness theorem.

1.3.5 Existence and uniqueness

- **Theorem: (*Existence and Uniqueness*)**

  If $f(x,y)$ and $f_y(x,y)$ are continuous functions of x and y in a region $|x - \bar{x}| < a$ and $|y - \bar{y}| < b$, where $a, b > 0$ then there is only one solution $y = y(x)$, defined in some interval $|x - \bar{x}| < h \leq a$, where $h > 0$, which satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \quad \text{with} \quad y(\bar{x}) = \bar{y}.$$

  (f_y denotes the partial derivative of f with respect to y.)

- **Higher orders:** The theorem can be extended in a straightforward way to an n^{th} order ODE when n independent constraints (in the form of initial conditions) are required to guarantee existence and uniqueness.

- **Points to note:**

  - Only local existence and uniqueness are guaranteed for initial value problems.
  - The theorem says nothing about global existence or about existence and uniqueness for boundary value problems.
  - The existence and uniqueness theorem does not work in reverse. That is there are initial value problems with unique solutions for which the conditions of the theorem are violated.

1.3.6 Existence and uniqueness for linear ODEs

The existence and uniqueness theorem for linear ODEs has a much stronger form.

- **Theorem: (*Existence and Uniqueness for Linear ODEs*)** If $p(x)$ and $q(x)$ are continuous functions on an interval I, if $\bar{x} \in I$ and if $\bar{y} \in \mathbb{R}$, then there exists a unique solution $y = y(x)$ throughout the interval I for the ODE

  $$y' + p(x)\, y = q(x)$$

  which also satisfies the initial condition

  $$y(\bar{x}) = \bar{y}.$$

  This solution is a differentiable function and it satisfies the ODE throughout I.

- **Higher orders:** The theorem can be extended in a straightforward way to an n^{th} order linear ODE when n independent constraints (in the form of initial conditions) are required to guarantee existence and uniqueness.

2 First-Order Ordinary Differential Equations

First-order ordinary differential equations describing $y(x)$ have the forms

$$F(x, y, y') = 0 \qquad \text{or} \qquad y' = f(x, y).$$

The first of these forms is more general, but where it can be solved for y', it can be written in the second form. In this section we shall consider a variety of ways in which solutions can be obtained for particular forms of first-order ODE.

2.1 Graphical approach

Before seeking any actual solutions, however, it can be noted that the ODE itself contains a lot of information about the nature of its solutions. This is because the equation

$$y' = f(x, y)$$

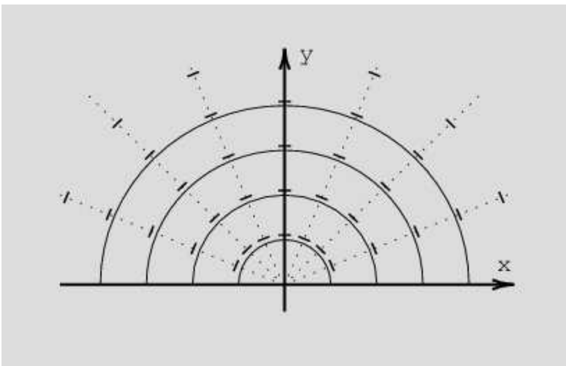
gives the slope of the function $y(x)$. In the plane of x and y it provides the direction in which the solution must be changing at any point (x, y). This is called the direction field of the ODE.

- **The direction field** of the ODE $y' = f(x, y)$ is the set of all direction vectors having the same direction as the vector $(1, y')$, at each point (x, y), in the plane of x and y.
- **Integral curves** are curves which are everywhere tangent to the direction field. Each integral curve represents a solution of the ODE.
- **Example 1:** If $y' = -x/y$, for $y(x) > 0$, then at any point (x, y) the direction in which the solution changes is $(1, -x/y)$.

  The direction does not change if it is multiplied by a scalar, so this is the same as the direction $(y, -x)$.

  At a point (x, y), the direction of $(y, -x)$ is at right angles to the line from the origin to (x, y), since $(x, y) \cdot (y, -x) = 0$. We can therefore easily sketch the direction field as short vectors at right angles to any line emerging from the origin.

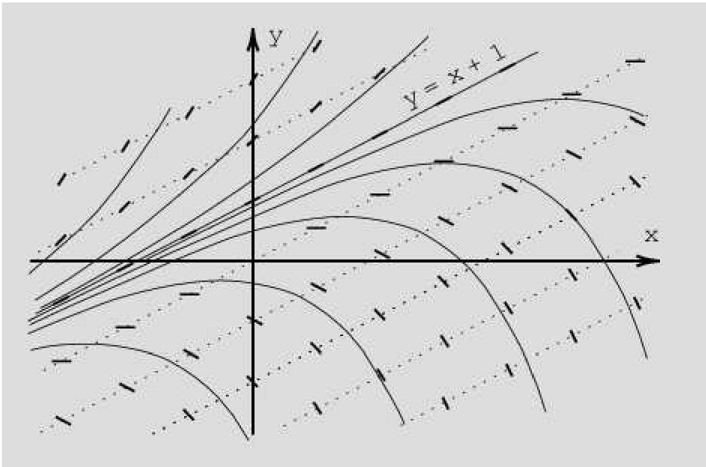
  It is clear that the integral curves in this example, which are tangent to the direction field at any point, must be semicircles in the half-plane $y > 0$, as indicated by the thin lines in the illustration.



A note on Existence and Uniqueness: Only one circle, with a fixed centre, can pass through any given point. Thus, if we specify some point, via a initial condition at which $y(\bar{x}) = \bar{y} > 0$, that must lie on a solution of $y' = -x/y$, this picks out a unique semicircle, with its centre at $(0, 0)$, as the solution that passes through $(\bar{x}, \bar{y})$.

More generally, for any first-order ODE $y' = f(x, y)$, if $f(x, y)$ and $f_y(x, y)$ are continuous, the direction field does not change direction abruptly. Intuitively, since there is only one direction in which the path can move at any point, there can be only one path, starting at some point $(\bar{x}, \bar{y})$, that is tangent to all of these smoothly changing direction vectors. It is then clear that only one integral curve (or solution) can pass through any given point $(\bar{x}, \bar{y})$.

- Direction fields are not always as easy to identify as in the example above. More generally, it is useful to identify isoclines and critical points as a means of building up a picture of the direction field:
  - **Isoclines** are paths in the space of x and y on which $y' = const$. Clearly, the directions in the direction field are the same along any one isocline.
  - **Critical points** arise where isoclines with different directions intersect.
  - **Example:** Considering the previous example $y' = -x/y$ again, in which each of the radii is an isocline, the origin $x = y = 0$ is a critical point.
- **Example 2:** If we consider $y' = y - x$, isoclines are given by the straight lines $y = x + c$, for a constant c, along which $y' = c$. The corresponding direction vector is thus $(1, c)$. Knowing this, the direction field and integral curves can be sketched fairly easily:



  Note that the path $y = x + 1$ has all of its direction vectors pointing in the same direction as the path, so that $y = x + 1$ is also an integral curve (a solution). It identifies a natural asymptote for other integral curves.

  There are no critical points for this ODE.

- The graphical approach helps to uncover qualitative behaviour of solutions of ordinary differential equations. Where quantitative results are needed, solutions must be found.

2.2 Solving first-order ODEs

It is not always possible to solve ordinary differential equations analytically. Even when the solution of an ODE is known to exist, it is not always possible to find the solution in terms of known standard functions, such as powers, exponentials, logarithms, trigonometric functions or even more specialised functions such as Bessel functions, error functions, etc.

Other methods, including numerical techniques, perturbation methods or graphical approaches, may need to be invoked to obtain or, if necessary, to approximate the solution. In very many cases, that cannot be solved in terms of standard functions, an infinite power series solution can be developed to provide the correct solution at least over some interval where the series converges, although more approximate solutions might sometimes be more revealing.

On the other hand, there are several forms of first-order ODE that can be solved analytically. There are also some particular ODEs which can be solved by using suitable transformations. We will now outline each of these types of equation and the ways in which they can be solved.

2.2.1 Separable first-order ODEs

- An ODE describing $y(x)$ is separable if it can be rearranged into the form

$$g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = h(x)$$

for some functions $g(\cdot)$ and $h(\cdot)$. Formally, multiplying both sides by $\mathrm{d}x$, this produces the form of the ODE

$$g(y)\,\mathrm{d}y = h(x)\,\mathrm{d}x$$

in which all dependence on x and $\mathrm{d}x$ has been separated onto one side of the equation with all dependence on y and $\mathrm{d}y$ on the other. In this form, each side of the equation can simply be integrated to provide a solution

$$\int g(y)\,\mathrm{d}y = \int h(x)\,\mathrm{d}x + A$$

for an arbitrary constant of integration A. It might not always be possible to solve the resulting formula explicitly for y, but, if that is the case, the formula does at least provide an implicit relationship between x and the solution y.

Example 1. The ODE $(1-y^2)\sin(x)y' - y\cos(x) = 0$ can be rearranged, through dividing by y and by $\sin(x)$, to give

$$\Big(\frac{1}{y} - y\Big)y' = \frac{\cos(x)}{\sin(x)}.$$

Integrating both sides therefore gives

$$\int \Big(\frac{1}{y} - y\Big)\mathrm{d}y = \int \frac{\cos(x)}{\sin(x)}\,\mathrm{d}x + A$$

so that

$$\ln|y| - \tfrac{1}{2}y^2 = \ln|\sin x| + A.$$

This result cannot be solved explicitly for the solution $y(x)$.

Example 2. The function $v(z)$ satisfies $\frac{\mathrm{d}v}{\mathrm{d}z} = -3z^2\,\mathrm{e}^v$ with $v(0) = 0$. Rearranging the ODE gives

$$\mathrm{e}^{-v}\frac{\mathrm{d}v}{\mathrm{d}z} = -3z^2$$

so that

$$\int \mathrm{e}^{-v}\,\mathrm{d}v = -3\int z^2\,\mathrm{d}z + A \quad \text{or} \quad -\mathrm{e}^{-v} = -z^3 + A$$

giving the solution

$$v(z) = -\ln(z^3 - A)$$

for an arbitrary constant A. Applying the initial condition $v(0) = 0$ requires

$$0 = -\ln(-A) \quad \text{so that} \quad A = -1.$$

The unique solution that satisfies the initial condition is therefore

$$v = -\ln(z^3 + 1).$$

2.2.2 First-order ODEs of homogeneous type

- An ODE for $y(x)$ is of homogeneous type if it can be rearranged into the form

$$y' = f(y/x)$$

and it can be simplified by using the substitution

$$z(x) = y(x)/x \quad \text{or} \quad y(x) = x\ z(x)$$

where $z(x)$ is another function of x. Differentiating gives

$$y' = xz' + z \quad \text{so that} \quad xz' + z = f(z) \quad \text{or} \quad z' = \frac{f(z) - z}{x}$$

which is a separable equation, that we already know how to solve for $z(x)$.

  - The transformation $z = y/x$ reduces an ODE of homogeneous type describing $y(x)$ to a separable ODE describing $z(x)$.
  - Once a solution for $z(x)$ is found, the solution for $y(x)$ is simply $y = x\ z(x)$.

Example. The ODE for $u(t)$

$$tu\frac{\mathrm{d}u}{\mathrm{d}t} = u^2 + 3t\sqrt{tu} \qquad \text{becomes} \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{u}{t} + 3\sqrt{t/u}$$

after dividing both sides by tu. The right hand side is a function of u/t so that the ODE is of homogeneous type. It can be turned into a separable ODE using the substitution $z(t) = u/t$ or

$$u = tz \quad \text{so that} \quad u' = tz' + z \quad \text{giving} \quad tz' + z = z + 3z^{-1/2}$$

Simplifying and separating variables gives

$$z' = 3z^{-1/2}/t \quad \text{or} \quad z^{1/2}z' = 3/t$$

and so integrating gives

$$\int z^{1/2}\,\mathrm{d}z = 3\int \frac{\mathrm{d}t}{t} + A \quad \text{or} \quad \tfrac{2}{3}z^{3/2} = 3\ln|t| + A$$

for an arbitrary constant A. Solving for z and substituting into $u = tz$ to find the solution u gives

$$z = \left(\tfrac{9}{2}\ln|t| + B\right)^{2/3} \quad \text{and so} \quad u = tz = t\left(\tfrac{9}{2}\ln|t| + B\right)^{2/3}$$

where B is an arbitrary constant.

Note. The constant $\frac{9}{2}A$ is arbitrary, so there is no need to write it as $\frac{9}{2}A$ and we might as well write it as one single symbol B.
There is no error made if we leave it as $\frac{9}{2}A$, but writing it as B is a little simpler.

2.2.3 Linear first-order ODEs

- Linear ODEs have the form
$$a(x)\frac{dy}{dx} + b(x)y = c(x),$$
where $a(x), b(x)$ and $c(x)$ are given functions.

- We divide the ODE by $a(x)$ to transform it into its *standard form*
$$\frac{dy}{dx} + p(x)y = q(x). \tag{1}$$

- This equation can be integrated by using the *integrating factor*
$$I(x) = e^{\int p(x)dx}.$$
Note that the constant of integration can be discarded when determining the integrating factor.

- After multiplying the standard form of the ODE (1) by the integrating factor, it can be rewritten in the form
$$\frac{d}{dx}\left(y\ I(x)\right) = q(x)I(x).$$

- This can easily be integrated to give
$$y(x) = \frac{1}{I(x)}\int q(x)\ I(x)\ dx$$

- This integration produces one constant of integration which has to be determined from the initial condition.

- Table 1 shows a step-by-step illustration of how this method works.

Table 1: The integration of a linear first-order ODE.

Step	General Procedure	Example
1. Identify the terms:	$a(x)\frac{dy}{dx} + b(x)y = c(x)$	$\underbrace{2x}_{a(x)}\frac{dy}{dx} + \underbrace{4x^2}_{b(x)}y = \underbrace{2x^2}_{c(x)}$
2. Transform into the standard form (i.e. divide by $a(x)$ if required):	$\frac{dy}{dx} + p(x)y = q(x)$	$\frac{dy}{dx} + \underbrace{2x}_{p(x)}y = \underbrace{x}_{q(x)}$
3. Determine the integrating factor (ignore the constant of integration):	$I(x) = e^{\int p(x)dx}$	$I(x) = e^{\int 2xdx} = e^{x^2}$
4. Multiply the ODE by $I(x)$ and rewrite the LHS:	$\frac{d}{dx}\left(y\ I(x)\right) = q(x)I(x)$	$\frac{d}{dx}\left(y\ e^{x^2}\right) = xe^{x^2}$
5. Integrate (don't forget the constant of integration!):	$y\ I(x) = \int q(x)I(x)dx$	$y\ e^{x^2} = \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + C$
6. Solve for $y(x)$:	$y(x) = \frac{1}{I(x)}\int q(x)I(x)dx$	$y(x) = \frac{1}{2} + Ce^{-x^2}$

- Again, the last step produces the *general solution.* The *specific solution* is determined by fixing the constant via the initial condition.

- **Note:** The initial condition cannot be placed where either $p(x)$ or $q(x)$ are singular. If the ODE is given in its general form, $a(x)y' + b(x)y = c(x)$, this situation arises at points where the coefficient multiplying the highest derivative, $a(x)$, vanishes. This is nearly always a "sign of trouble" and we will encounter other examples throughout this course.

- The solution of a linear first-order ODEs can be written as

$$y(x) = y_P(x) + y_H(x)$$

where $y_P(x)$ is *any* particular solution of the ODE $y' + p(x)\,y = q(x)$, and $y_H(x)$ is the *general* solution of the corresponding *homogenous ODE*[2] $y' + p(x)\,y = 0$. This is a generic feature of all linear ODEs, not just linear ODEs of first order.

[2]Do not confuse the "ODE of homogenous type", discussed in section 2.2.2, with the *homogenous linear ODE*!

3 Second-Order Ordinary Differential Equations

The general form of a second-order ODE is given by

$$\mathcal{F}\big(x, y(x), y'(x), y''(x)\big) = 0.$$

It is typically augmented by two boundary or initial conditions, i.e constraints of the form

$$y(X) = Y, \quad y'(X) = Z,$$

or

$$y(X_1) = Y_1, \quad y(X_2) = Y_2$$

where the constants X, Y, Z (or X_1, Y_1, X_2, Y_2) are given.

Often the ODE can be written in explicit form as

$$y''(x) = f\big(x, y(x), y'(x)\big).$$

In this lecture we will mainly concentrate on linear second-order ODEs. (In section 3.3 we will briefly discuss the solution of two particular types of nonlinear ODEs).

3.1 Some theory for linear second-order ODEs

- In general, we shall write a *linear* second-order ODE for $y(x)$ in one of two ways, either as

  $$a(x)\, y'' + b(x)\, y' + c(x)\, y = d(x)$$

  or as

  $$y'' + p(x)\, y' + q\,(x) y = r(x).$$

  We will take these ODEs to be defined on an interval

  $$I = (\alpha, \beta) = \{\, x \mid \alpha < x < \beta \,\}$$

  which is chosen such that, at all values of x in I:

  - $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are defined and continuous
  - and $a(x)$ is never zero

  so that the functions $p(x)$, $q(x)$ and $r(x)$, which are defined as

  $$p(x) = b(x)/a(x), \quad q(x) = c(x)/a(x), \quad r(x) = d(x)/a(x),$$

  are also defined and continuous throughout I.

- **Theorem (*Existence and Uniqueness*):** If $y(x)$ satisfies the ODE $y'' + p(x)\, y' + q(x)\, y = r(x)$, and the functions $p(x)$, $q(x)$ and $r(x)$ are continuous throughout the interval I, then there is only one solution that satisfies the pair of initial conditions

  $$y(X) = Y \quad \text{and} \quad y'(X) = Z$$

  and this solution exists throughout the interval I.

  This theorem guarantees that solutions will exist throughout the interval I and that the two initial conditions, one giving the value of y and the other giving the value of its derivative y', both specified at the same point in I, are enough to select a unique solution.

  Note that the existence and uniqueness theorem only applies to initial value problems!

- **Superposition:** In the special case in which the ODE has $r(x)$ set equal to zero, that is for the special form of the ODE

  $$y'' + p(x)\, y' + q(x)\, y = 0$$

  which is known as the 'homogeneous' form of the ODE, a linear combination of any solutions is also a solution. Thus if $y_1(x)$ and $y_2(x)$ are solutions of $y'' + p(x)\, y' + q(x)\, y = 0$ then so is any function that can be written as $A\, y_1(x) + B\, y_2(x)$ for any constants A and B.

- **Fundamental Solutions:** What is more, *any solution* of the homogeneous second-order linear ODE $y'' + p(x)\,y' + q(x)\,y = 0$ can be written as a linear combination of only two solutions $y_1(x)$ and $y_2(x)$, known as 'fundamental solutions,' provided $y_1(x)$ and $y_2(x)$ are nonzero and linearly independent.

  [Reminder: Two functions $y_1(x)$ and $y_2(x)$, defined on I, are said to be ***linearly independent*** on I if the only linear combination of them that adds up to zero, so that $A\,y_1(x) + B\,y_2(x) = 0$ for all $x \in I$, is the one for which $A = B = 0$.]

  The choice of fundamental solutions is not unique. For instance, if $\{y_1(x), y_2(x)\}$ is a set of fundamental solutions for a given linear homogeneous ODE then $\big\{y_1(x), \big(y_1(x) + y_2(x)\big)\big\}$ is another set of fundamental solutions.

  A solution of the homogeneous ODE is sometimes called a *complementary function.*

- **General Solutions:** Any solution of the non-homogeneous ODE $y'' + p(x)\,y' + q(x)\,y = r(x)$ has the form, known as the 'general solution'

$$y = y_P(x) + A\,y_1(x) + B\,y_2(x)$$

  where $y_P(x)$, known as a 'particular solution,' is a solution of the non-homogeneous ODE, and $y_1(x)$ and $y_2(x)$ are fundamental solutions of the homogeneous form of the ODE, in which $r(x)$ is set to zero.

- The solution to a specific boundary or initial value problem can therefore be obtained in four steps:

  1. Find the general solutions of the homogeneous ODE:

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad \Longrightarrow \quad y_H(x) = A\,y_1(x) + B\,y_2(x),$$

     where $y_1(x)$ and $y_2(x)$ are two nonzero, linearly independent solutions, i.e. they are fundamental solutions of the homogenous ODE.
  2. Find a particular solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \quad \Longrightarrow \quad y_P(x).$$

  3. Write down the general solution

$$y(x) = y_P(x) + y_H(x) = y_P(x) + A\,y_1(x) + B\,y_2(x).$$

  4. Determine the constants A and B from the boundary or initial conditions.

3.2 Linear second-order ODEs with constant coefficients

3.2.1 The general solution of the homogenous ODE

- Second-order ODEs for $y(x)$ of the form

$$y'' + py' + qy = 0 \quad \text{with } p \text{ and } q \text{ constant}$$

  can always be solved, for all real values of x, using the ansatz

$$y = e^{\lambda x}.$$

  [***Important*:** The method does not generally work when p and q are not constant.]

- Inserting $y = e^{\lambda x}$ into the ODE and cancelling the common factor $e^{\lambda x}$ yields the so-called *characteristic polynomial*

$$\lambda^2 + p\lambda + q = 0 \quad \text{with roots} \quad \lambda = \tfrac{1}{2}\Big(-p \pm \sqrt{p^2 - 4q}\Big).$$

  The roots, and hence the nature of the solutions, depends on the sign of the 'discriminant' $p^2 - 4q$:

Case 1: $p^2 - 4q > 0$

If the discriminant is positive ($p^2 - 4q > 0$) then λ has two distinct real roots of the form

$$\lambda_1 = \tfrac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right) \quad \text{and} \quad \lambda_2 = \tfrac{1}{2}\left(-p + \sqrt{p^2 - 4q}\right).$$

The general solution of the homogenous ODE can therefore be written as

$$y = A\,e^{\lambda_1 x} + B\,e^{\lambda_2 x},$$

where A and B are arbitrary constants.

Case 2: $p^2 - 4q < 0$

If the discriminant is negative ($p^2 - 4q < 0$) then λ has two complex conjugate roots of the form

$$\lambda = \mu \pm i\omega \quad \text{with} \quad \mu = -\tfrac{1}{2}p \quad \text{and} \quad \omega = \tfrac{1}{2}\sqrt{4q - p^2}.$$

The general solution of the homogeneous ODE can then be written as

$$y = A\,e^{\mu x}\cos(\omega x) + B\,e^{\mu x}\sin(\omega x),$$

where A and B are arbitrary constants.

Case 3: $p^2 - 4q = 0$

If the discriminant is zero ($p^2 - 4q = 0$) then the characteristic polynomial has one double root

$$\lambda_{1,2} = \lambda = -\tfrac{1}{2}p$$

giving only one fundamental solution $y_1 = e^{\lambda x} = e^{-px/2}$. However another fundamental solution is $y_2 = x\,e^{\lambda x} = x\,e^{-px/2}$ (*Exercise*: check this by substitution). The general solution of the homogeneous ODE can therefore be written as

$$y = A\,e^{-px/2} + B\,x\,e^{-px/2},$$

where A and B are arbitrary constants.

3.2.2 The particular solution of the inhomogenous ODE: The method of undetermined coefficients

- The method of undetermined coefficients is, more or less, a process of trial and error, or guesswork, based on making a suitable initial assumption about the overall form of the solution.
- The method and its pitfalls are best illustrated with an example:

$$y'' + py' + qy = A\,e^{ax}.$$

Initial ansatz:

Given that the RHS e^{ax} retains its functional form when differentiated, it is tempting to try a solution in the form $y = C\,e^{ax}$, having $y' = Ca\,e^{ax}$ and $y'' = Ca^2\,e^{ax}$, so that

$$Ca^2\,e^{ax} + pCa\,e^{ax} + qC\,e^{ax} = A\,e^{ax} \quad \text{or} \quad (a^2 + pa + q)C = A$$

which requires that $C = \frac{A}{a^2+pa+q}$, leading to the particular solution

$$y = y_{\mathrm{p}}(x) = \frac{A}{a^2 + pa + q}\,e^{ax} \quad \text{provided} \quad a^2 + pa + q \neq 0.$$

Modification if a is a (single) root of the characteristic polynomial

If $a^2 + pa + q = 0$ the initial ansatz, that $y = C\,e^{ax}$, is obviously inadequate. We note that this case arises if the a happens to be a root of the characteristic polynomial of the associated homogeneous ODE. In this case, another ansatz is appropriate. We assume, instead, that

$$y = Cx\,e^{ax} \quad \text{so that} \quad y' = C(1 + ax)\,e^{ax}, \quad y'' = C(2a + a^2x)\,e^{ax}.$$

In this case the ODE gives

$$C(2a + a^2x)\,e^{ax} + pC(1 + ax)\,e^{ax} + qCx\,e^{ax} = A\,e^{ax}$$

or

$$\big(x\,\underbrace{(a^2 + ap + q)}_{=0} + 2a + p\big)C = (2a + p)C = A$$

since $a^2 + ap + q = 0$. Thus we find that $C = \frac{A}{2a+p}$, leading to the particular solution

$$y = y_\mathrm{p}(x) = \frac{A}{2a + p}\,x\,e^{ax} \quad \text{provided} \quad a^2 + pa + q = 0 \quad \text{and} \quad 2a + p \neq 0.$$

Modification if a is a double root of the characteristic polynomial

If both $a^2 + pa + q$ and $2a + p$ are zero, then both guesses, that $y = C\,e^{ax}$ or $y = Cx\,e^{ax}$, are obviously inadequate. We note that this case arises if a is a double root of the characteristic polynomial. In this case, yet another ansatz is appropriate. We now assume that

$$y = Cx^2\,e^{ax} \quad \text{so that} \quad y' = C(2x + ax^2)\,e^{ax}, \quad y'' = C(2 + 4ax + a^2x^2)\,e^{ax}.$$

In this case the ODE gives

$$C(2 + 4ax + a^2x^2)\,e^{ax} + pC(2x + ax^2)\,e^{ax} + qCx^2\,e^{ax} = A\,e^{ax}$$

or

$$\big(x^2\,\underbrace{(a^2 + ap + q)}_{=0} + x\,\underbrace{2(2a + p)}_{=0} + 2\big)C = 2C = A$$

since $a^2 + ap + q = 0$ and $2a + p = 0$. Thus we find that $C = \frac{1}{2}A$, leading to the particular solution

$$y = y_\mathrm{p}(x) = \tfrac{1}{2}A\,x^2\,e^{ax} \quad \text{provided} \quad a^2 + pa + q = 0 \quad \text{and} \quad 2a + p = 0.$$

- This example shows that a particular solution of the ODE $y'' + py' + qy = A\,e^{ax}$, with constant coefficients p and q, typically takes the form $Cx^m\,e^{ax}$ for an integer power m that depends on whether or not e^{ax} and $x\,e^{ax}$ are solutions of the homogeneous equation.
- Based on this observation we can formulate the "method of undetermined coefficients" for inhomogenous, constant-coefficient of the form

$$y'' + p\,y' + q\,y = A_1\,r_1(x) + A_2\,r_2(x) + \cdots + A_n\,r_n(x)$$

where the RHS is a linear combination of n given, linearly-independent functions $r_i(x)$ $(i = 1, ..., n)$.

The idea is the following:

1. We initially try to find a particular solution that contains the same (linearly independent) functions that occur on the RHS:

$$y_P^{[initial]}(x) = C_1\,r_1(x) + C_2\,r_2(x) + \cdots + C_n\,r_n(x)$$

with undetermined (constant) coefficients C_i $(i = 1, ..., n)$. The plan is to insert this into the ODE and to collect the coefficients that multiply the same functions $r_i(x)$ $(i = 1, ..., n)$. Since the $r_i(x)$ are linearly independent, their linear combination can only vanish if the coefficients multiplying them vanish individually. This provides n equations for the n unknown coefficients C_i $(i = 1, ..., n)$. Bingo!

2. This doesn't work, however, if the derivative of any of the $r_i(x)$ cannot be expressed as a linear combination of the terms in $y_P^{[initial]}$. [In the above example, the derivatives of $r_1(x) = e^{ax}$ were simply multiples of e^{ax}, so no additional functions arose. However, if $r_1(x) = x^2$, say, the differentiation of $y_P^{[initial]}$ would also produce $r_1'(x) = 2x$ and $r_1''(x) = 2$.]
To deal with such cases, we generalise our ansatz to the form

$$\begin{aligned} y_P^{[better]}(x) &= C_1\,r_1(x) + C_2\,r_2(x) + \cdots + C_n\,r_n(x) \\ &+ D_1\,r_1'(x) + D_2\,r_2'(x) + \cdots + D_n\,r_n'(x) \\ &+ E_1\,r_1''(x) + E_2\,r_2''(x) + \cdots + E_n\,r_n''(x), \end{aligned}$$

where we set the coefficients E_i and D_i $(i = 1, ..., n)$ that multiply terms that are already contained in $y_P^{[initial]}(x)$ to zero.

3. Finally, we have to deal with the case where some of the terms in $y_P^{[better]}$ are solutions of the homogenous ODE $y'' + p\,y' + q\,y = 0$. Let $\tilde{r}(x)$ be a term in $y_P^{[better]}(x)$ that is a solution of the homogeneous ODE. For each such term, we replace $\tilde{r}(x)$ by $x^m\,\tilde{r}(x)$, where m is the smallest positive integer for which $x^m\,\tilde{r}(x)$ does not solve the homogeneous ODE. If the derivatives of $x^m\,\tilde{r}(x)$ create new linearly independent functions, not yet contained in $y_P^{[better]}$, add these too.

3.3 Some nonlinear second-order ODEs

In a few cases, second-order ODEs can be solved as first-order ODEs. Two important cases are those that take the form

$$\tfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} = f\big(y, \tfrac{\mathrm{d}y}{\mathrm{d}t}\big) \qquad \text{or} \qquad \tfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} = f\big(t, \tfrac{\mathrm{d}y}{\mathrm{d}t}\big)$$

when describing $y(t)$. The first of these represents second-order ODEs that are autonomous, which is to say that they do not depend on t (apart from differentiating with respect to t). The second represents second-order ODEs that do not depend on y (except as derivatives of y).

3.3.1 Second-order ODEs for $y(t)$ that do not depend on y

Such ODEs take the form

$$y'' = f(t, y').$$

All we need to do is note that this is actually a first-order ODE for $y'(t)$. If we write, $v(t) = y'(t)$ then the ODE is clearly a first-order ODE for v, namely

$$v' = f\big(t, v\big).$$

If this is solved to find a solution $v(t)$, then $y(t)$ is a solution of the first-order ODE $y' = v(t)$.

3.3.2 Autonomous second-order ODEs

Autonomous second-order ODEs which, when describing $y(t)$ have the form

$$y'' = f(y, y')$$

can also be solved by writing $v = y'(t)$, but in a different way. Differentiating $y'(t) = v$ gives

$$y'' = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}y}.$$

The ODE can therefore be rewritten in the form

$$v\tfrac{\mathrm{d}v}{\mathrm{d}y} = f(y, v)$$

which, if we think of v as being a function of y, is a first-order ODE for v. If we can solve for $v = v(y)$ then $y(t)$ is a solution of the first-order ODE $y' = v(y)$.

4 Mechanics applications of second-order ODEs

- Second-order linear ODEs with constant coefficients arise in many physical applications. One physical systems whose behaviour is governed by an ODE of the form

$$m\,\ddot{x} + k\,\dot{x} + c\,x = f(t) \tag{2}$$

is the damped mechanical oscillator shown in Fig. 1. In this case, m represents the mass of the particle attached to the spring, k is a measure of the strength of the damper, and c represents the spring stiffness. $f(t)$ is the applied external force. t represents time and $x(t)$ is the displacement of the mass from its rest position.

- The physical interpretation of the individual terms in equation (2) is as follows. The term $k\dot{x}$ represents the force exerted by the damper on the mass: for a linear damper, this force is proportional to the velocity $\dot{x}$ and it resists the motion. The term cx represents the force exerted by the spring on the mass: for a linear spring, this force is proportional to the displacement x and it acts in the direction opposite to the displacement. Equation (2) expresses Newton's law, which states that the sum of all forces acting on the particle is equal to its mass times its acceleration, $m\ddot{x}$.

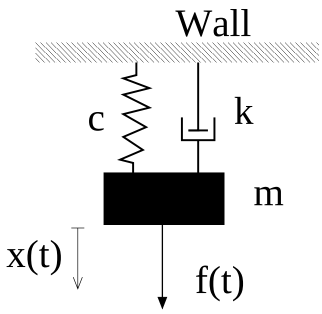


Figure 1: Sketch of a damped mechanical oscillator consisting of a mass m attached to a rigid wall by a linear spring of spring stiffness c and a damper with damping constant k. A time-dependent force, $f(t)$, is applied to the mass and the displacement of the mass from its rest position is represented by $x(t)$.

- The initial conditions are given by the initial position,

$$x(t = 0) = x_0,$$

and the initial velocity of the mass at time $t = 0$,

$$\left.\frac{dx}{dt}\right|_{t=0} = v_0.$$

4.1 The unforced case: $f(t) = 0$ – Eigenfrequencies

- If $f(t) = 0$, equation (2) is reduced to its *homogeneous* form,

$$m\ddot{x} + k\dot{x} + cx = 0. \tag{3}$$

which describes the motion of the mass in absence of any external forcing.

- To reduce the number of parameters required to classificy the character of the particle's motion, we re-write the ODE (3) as

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = 0, \tag{4}$$

where

$$\delta = \frac{k}{2m} \quad \text{and} \quad \omega^2 = \frac{c}{m}. \tag{5}$$

- Note that in the damped mechanical oscillator the coefficients m, k and c are positive. Hence δ and ω^2 are positive, too.

- The solution to the corresponding characteristic equation
$$\lambda^2 + 2\delta\lambda + \omega^2 = 0, \tag{6}$$
is given by
$$\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega^2} \tag{7}$$
or, written in a different form,
$$\lambda_{1,2} = -\delta \pm i\sqrt{\omega^2 - \delta^2}. \tag{8}$$
- This allows a straightforward identification of four distinct types of motion:

I. Purely damped motion: $\delta > \omega$

In this case equation (7) shows that both roots are real and the solution is given by
$$x(t) = Ae^{(-\delta+\sqrt{\delta^2-\omega^2})t} + Be^{(-\delta-\sqrt{\delta^2-\omega^2})t}. \tag{9}$$
Since both exponents are negative, this represents a *purely damped motion*, i.e. the system does not perform any oscillations.

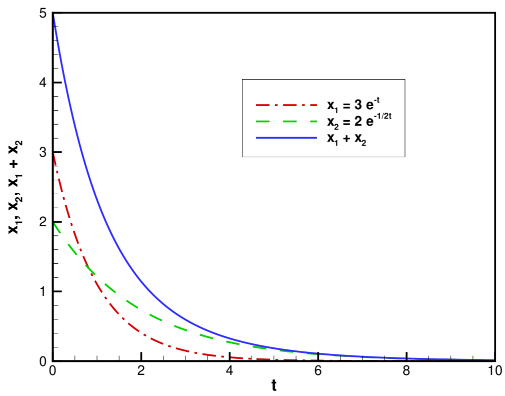


Figure 2: Illustration of a purely damped motion. The mass approaches its equilibrium position $x = 0$ monotonically.

II. Critically damped motion: $\delta = \omega$

In this case the square root in (7) vanishes and we have $\lambda_1 = \lambda_2 = -\delta$ and the general solution is given by
$$x(t) = Ae^{-\delta t} + Bte^{-\delta t}. \tag{10}$$
This represents a *critically damped motion* in which $x(t)$ approaches zero but can cross the value $x = 0$ (at most) once (when $t = -A/B$; it depends on the initial conditions which determine A and B, if this happens for $t > 0$).

III. Damped oscillation: $\delta < \omega$

In this case (8) shows that both roots are complex. The general solution is then given by
$$x(t) = e^{-\delta t}\left(A\cos(t\sqrt{\omega^2-\delta^2}) + B\sin(t\sqrt{\omega^2-\delta^2})\right). \tag{11}$$
This solution represents a *damped oscillation* with frequency $\sqrt{\omega^2 - \delta^2}$ whose amplitude decays exponentially.

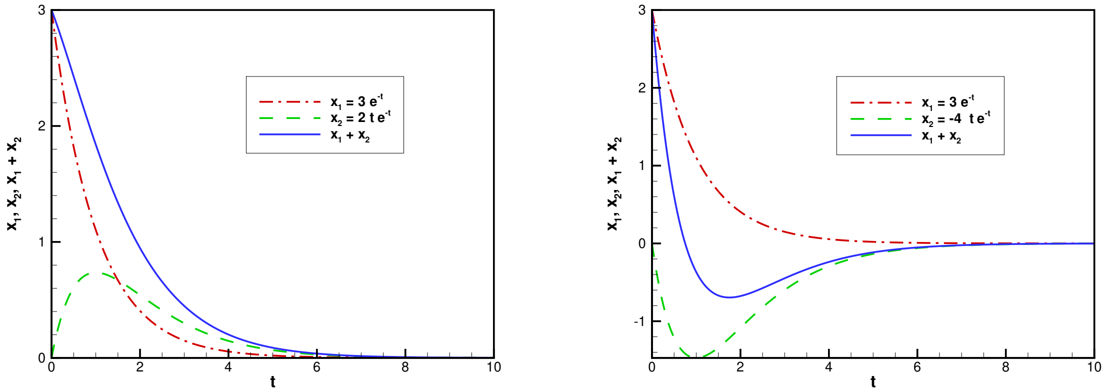


Figure 3: Illustration of critically damped motions. The mass approaches its equilibrium position, $x = 0$, with at most one "overshoot".

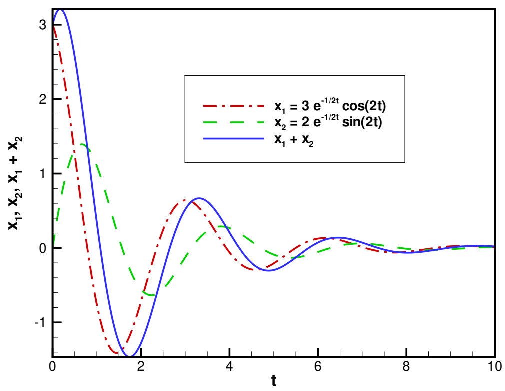


Figure 4: Illustration of a damped oscillation. The mass oscillates about its equilibrium position $x = 0$ and the amplitude of the oscillations decays exponentially.

IV: Undamped oscillations $\delta = 0$

The solution (11) is still valid and for $\delta = 0$ we obtain

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \tag{12}$$

which is an *undamped oscillatory motion* with *eigenfrequency* ω.

4.2 The inhomogeneous equation – Periodic forcing and resonance

- The general solution of the inhomogeneous equation

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = F(t), \tag{13}$$

where $F(t) = f(t)/m$ is given by

$$x(t) = x_H(t) + x_P(t). \tag{14}$$

- In (14), $x_H(t)$ is the general solution of the corresponding homogeneous equation (4) and thus contains the two free constants required to fulfill the initial conditions. $x_P(t)$ (the particular solution) is *any* solution of the inhomogeneous equation.

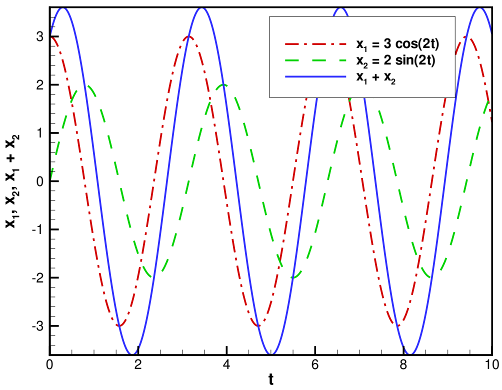


Figure 5: Illustration of an undamped oscillation. The mass performs harmonic oscillations about its equilibrium position $x = 0$.

- The most important form of forcing in many engineering applications is the harmonic forcing in which $F(t)$ has the form

$$F(t) = F\sin(\Omega t) \quad \text{or} \quad F(t) = F\cos(\Omega t).$$

- It is advantageous to carry out the calculation with complex numbers by writing

$$F(t) = Fe^{i\Omega t} = F\cos(\Omega t) + iF\sin(\Omega t). \tag{15}$$

  Once the calculation is completed we can extract the relevant real solution (corresponding to the cos or sin forcing) by taking the real or imaginary parts of the complex solution.

- We know from the previous section that we can find a solution for exponential forcing in the form

$$x_P(t) = Xe^{i\Omega t}. \tag{16}$$

  Hence we insert (16) and (15) into (13) and carry out the differentiations.

- After cancelling the common factor $e^{i\Omega t}$, this provides the following equation for the unknown amplitude X (which is complex in general):

$$X = \frac{F}{(\omega^2 - \Omega^2) + i\ (2\delta\Omega)} = F\ \frac{(\omega^2 - \Omega^2) - i\ (2\delta\Omega)}{(\omega^2 - \Omega^2)^2 + (2\delta\Omega)^2} \tag{17}$$

- Now we can multiply out $x_P(t) = Xe^{i\Omega t} = X(\cos(\Omega t) + i\sin(\Omega t))$ and extract the real or imaginary part to obtain the relevant real solution, i.e.

$$x_P(t) = \mathcal{R}(X)\cos(\Omega t) - \mathcal{I}(X)\sin(\Omega t) \qquad \text{for} \qquad F(t) = F\cos(\Omega t)$$

  and

$$x_P(t) = \mathcal{R}(X)\sin(\Omega t) + \mathcal{I}(X)\cos(\Omega t) \qquad \text{for} \qquad F(t) = F\sin(\Omega t).$$

- Alternatively, we can re-write (17) in polar form and thus obtain the *amplitude* of the response

$$|X| = \frac{F}{\sqrt{(\omega^2 - \Omega^2)^2 + (2\delta\Omega)^2}} = \frac{F/\omega^2}{\sqrt{\left(1 - (\Omega/\omega)^2\right)^2 + \left(2\,(\delta/\omega)\,(\Omega/\omega)\right)^2}}$$

  and the *phase angle*

$$\varphi = \arg(X) = \arctan\left(\frac{-2\delta\Omega}{\omega^2 - \Omega^2}\right). \tag{18}$$

- Hence, the particular solution can also be written as

$$x_P(t) = |X|\ e^{i(\Omega t+\varphi)}.$$

The relevant real solutions are again obtained by taking the real and imaginary part of this complex solution, i.e.

$$x_P(t) = |X|\cos(\Omega t+\varphi) \qquad \text{for} \qquad F(t) = F\cos(\Omega t)$$

and

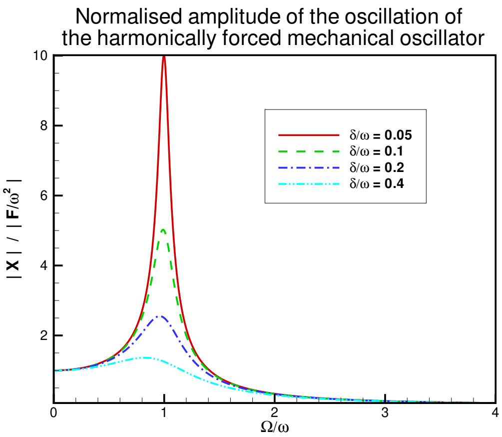
$$x_P(t) = |X|\sin(\Omega t+\varphi) \qquad \text{for} \qquad F(t) = F\sin(\Omega t).$$

Comparing this to (15) shows that the particular solution (i.e. the forced motion), $x_P(t)$, has a phase difference of φ against the forcing, $F(t)$.

- Note that for weak damping (small δ), the amplitude of the response, $|X|$, becomes very large when the excitation frequency Ω is close to the eigenfrequency of the system, $\Omega \approx \omega$. This is known as *resonance*. For vanishing damping, $\delta = 0$, the response becomes unbounded when $\Omega \to \omega$. This is the 'resonance catastrophe'.

In practical applications there is always some damping but the amplitudes of the forced oscillations of many physical systems can still become so large that the system is destroyed when the excitation frequency is sufficiently close to the eigenfrequency of the system. [Remember the wobbling Millenium Bridge?]

Here is a plot of the (normalised) amplitude of the oscillation as a function of the excitation frequency (normalised by the system's eigenfrequency ω) for various values of the damping parameter δ:



- Finally, note that for positive damping ($\delta > 0$), the homogeneous solution $x_H(t)$ decays rapidly with increasing time, i.e. $x_H(t) \to 0$. For sufficiently large times, only the forced motion $x_P(t)$ persists. Therefore, the solution $x_H(t)$ is often referred to as the *transient solution*.

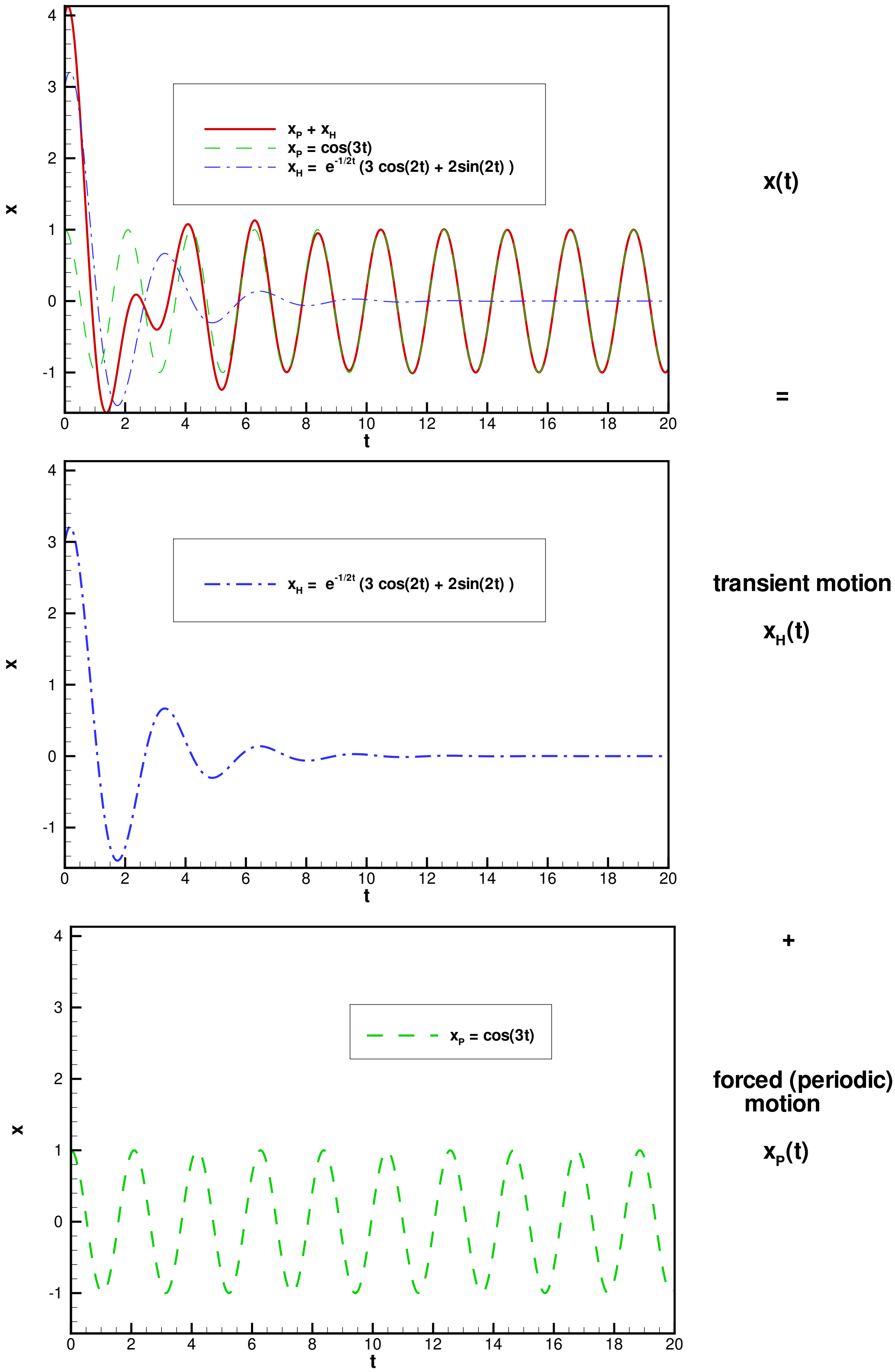


Figure 6: The displacement of a harmonically-forced, damped mechanical oscillator comprises the periodic (forced) solution $x_P(t)$ and the transient solution $x_H(t)$.