

# MATH10222: SOLUTIONS <sup>1</sup> V

## 0. Mechanics-free substitute for question 1

(a) The homogeneous solution of the ODE is given by

$$x_H(t) = B \cos(\omega t) + C \sin(\omega t)$$

for arbitrary constants  $B$  and  $C$ . [**Note:** By now you should be so familiar with this best-known of all 2nd-order ODEs, that you can simply write down this solution, rather than having to derive it via the characteristic polynomial!] Since  $\Omega \neq \omega$  an ansatz of the form  $x_P(t) = E \sin(\Omega t)$ ;  $\ddot{x}_P(t) = -E\Omega^2 \sin(\Omega t)$  for the particular solution should work since all terms in the ODE will vary with  $\sin(\Omega t)$ . Indeed, inserting the ansatz into the ODE yields

$$E \sin(\Omega t) (-\Omega^2 + \omega^2) = A\omega^2 \sin(\Omega t),$$

so

$$E = A \frac{\omega^2}{\omega^2 - \Omega^2} = \frac{A}{1 - (\Omega/\omega)^2}.$$

The general solution is therefore given by

$$x(t) = B \cos(\omega t) + C \sin(\omega t) + \frac{A}{1 - (\Omega/\omega)^2} \sin(\Omega t).$$

Applying the initial condition  $x(0) = 0$  yields  $B = 0$ ; applying  $\dot{x}(0) = 0$  gives the final result

$$x(t) = \frac{A}{1 - (\Omega/\omega)^2} \left[ \sin(\Omega t) - \left( \frac{\Omega}{\omega} \right) \sin(\omega t) \right].$$

$x(t)$  contains two periodic oscillations (about  $x = 0$ ) with frequencies  $\omega$  and  $\Omega$ . Note how the amplitude of the oscillation grows as  $\Omega \rightarrow \omega$ . Since  $|\sin(\Omega t) - (\Omega/\omega) \sin(\omega t)| < (1 + \Omega/\omega)$ , we can ensure that  $|x(t)| < H$  by choosing

$$A < H \frac{1 - (\Omega/\omega)^2}{1 + (\Omega/\omega)} = H \left( 1 - \left( \frac{\Omega}{\omega} \right) \right).$$

This shows that if we're close to resonance,  $\Omega \approx \omega$ , we can only apply a very small forcing amplitude  $A$  if we want to limit the amplitude of the oscillation to a given value.

(b) If  $\Omega = \omega$ , the RHS  $r(t) = \omega^2 A \sin(\Omega t)$  is a solution of the homogeneous ODE, therefore we have to multiply the naive ansatz for  $x_P$ , used in the previous question, by the smallest integer power of the independent variable  $t$  for which it ceases to be a solution of the homogeneous ODE. Multiplication by  $t$  does the trick, so we now modify the ansatz to

$$x_P(t) = E t \sin(\omega t).$$

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As observed on a previous example sheet, trigonometric RHSs are best done by performing the calculation in complex variables. Therefore, we re-write the ODE as

$$\ddot{x} + \omega^2 x = A \omega^2 e^{i\omega t},$$

and only use the imaginary part of the solution. Differentiating the “complexified” ansatz

$$x_P(t) = E t e^{i\omega t}$$

yields

$$\begin{aligned} \dot{x}_P(t) &= E (1 + i\omega t) e^{i\omega t}, \\ \ddot{x}_P(t) &= E (2i\omega - \omega^2 t) e^{i\omega t}. \end{aligned}$$

Insert into the ODE

$$E e^{i\omega t} (2i\omega - \omega^2 t + \omega^2 t) = A \omega^2 e^{i\omega t} \implies E = \frac{A \omega^2}{2i\omega} = -i \frac{A\omega}{2}.$$

Thus

$$x_P(t) = -i \frac{A\omega}{2} t e^{i\omega t} = -i \frac{A\omega}{2} t (\cos(\omega t) + i \sin(\omega t)).$$

Extracting the imaginary part of this complex solution <sup>2</sup> and adding the (unchanged) solution of the homogeneous ODE yields the general solution

$$x(t) = B \cos(\omega t) + C \sin(\omega t) - \frac{A\omega}{2} t \cos(\omega t)$$

for arbitrary constants  $B$  and  $C$ . Applying the initial conditions yields

$$x(t) = \frac{A}{2} (\sin(\omega t) - \omega t \cos(\omega t)).$$

$t_{shatter}$  is determined by the condition  $x(t_{shatter}) = -H$ , i.e.

$$-H = \frac{A}{2} (\sin(\omega t_{shatter}) - \omega t_{shatter} \cos(\omega t_{shatter})). \quad (1)$$

- (c) Equation (1) cannot be solved in closed form but zooming in on a plot of  $x(t) + H$ , for the parameter values specified in the question shows that  $t_{shatter} \approx 95.9673$ .

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<sup>2</sup>Note that if we had “stayed real”, differentiation of the ansatz  $x_p = E t \sin(\omega t)$  would have created cos-terms; inclusion of these (again pre-multiplied by  $t$  since  $\cos(\omega t)$  solves the homogeneous ODE) would have required the yet-again modified ansatz:  $x_p = E t \sin(\omega t) + F t \cos(\omega t)$ . After a considerable amount of algebra we would then have found that  $E = 0$ , so the original form of the ansatz wasn’t required...

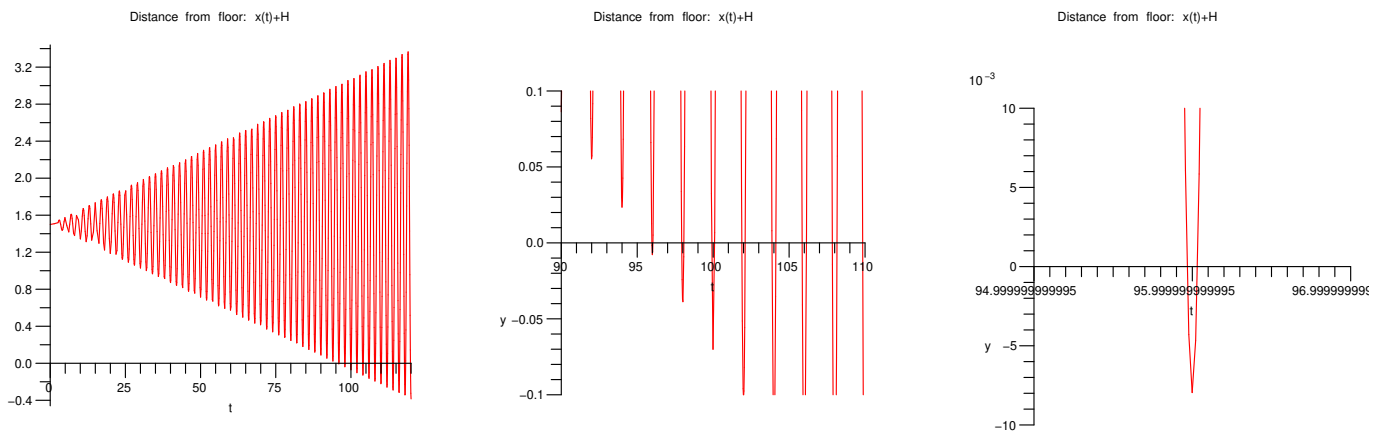


Figure 1: Zooming into a plot of  $x(t) + H$ , shows that  $t_{\text{shatter}} \approx 95.67$  for  $A = 0.01$ ,  $H = 1.5$ ,  $\omega = \pi$ .

1. Applications of second-order ODEs: “Resonance” or “How to destroy a coffee mug...”

(a) Consider the mug at an arbitrary time  $t$  when its handle is located at  $x(t)$  while the upper end of the spring has been moved to  $\hat{x}(t) = L + A \sin(\Omega t)$ . The instantaneous length of the spring is  $l(t) = \hat{x}(t) - x(t) = L + A \sin(\Omega t) - x(t)$  so its change in length (relative to the initial configuration in which the system is at rest) is  $\Delta l = l(t) - L = A \sin(\Omega t) - x(t)$ . The force that the spring exerts onto the mug is therefore given by  $c\Delta l = c(A \sin(\Omega t) - x(t))$  (positive if the force acts upwards). Inserting this into Newton’s law (mass times acceleration is equal to the sum of all forces acting on the mug) yields

$$m\ddot{x} = c(A \sin(\Omega t) - x(t)),$$

or

$$m \ddot{x} + c x = cA \sin(\Omega t),$$

as required.

(b) At  $t = 0$  the mug is in its initial position and at rest so the initial conditions are

$$x(0) = 0 \quad \text{and} \quad \left. \frac{dx}{dt} \right|_{t=0} = 0.$$

Before solving the IVP, we rewrite the ODE in its standard form

$$\ddot{x} + \omega^2 x = \frac{cA}{m} \sin(\Omega t) = A \omega^2 \sin(\Omega t). \tag{2}$$

The homogeneous solution of this ODE is given by

$$x_H(t) = B \cos(\omega t) + C \sin(\omega t)$$

for arbitrary constants  $B$  and  $C$ . [**Note:** By now you should be so familiar with this best-known of all 2nd-order ODEs, that you can simply write down this solution, rather than having to derive it via the characteristic polynomial!]

Since  $\Omega \neq \omega$  and there is no damping, an ansatz of the form  $x_P(t) = E \sin(\Omega t)$ ;  $\ddot{x}_P(t) = -E\Omega^2 \sin(\Omega t)$  for the particular solution should work since all terms in (2) will vary with  $\sin(\Omega t)$ . Indeed, inserting the ansatz into the ODE yields

$$E \sin(\Omega t)(-\Omega^2 + \omega^2) = A \omega^2 \sin(\Omega t),$$

so

$$E = A \frac{\omega^2}{\omega^2 - \Omega^2} = \frac{A}{1 - (\Omega/\omega)^2}.$$

The general solution is therefore given by

$$x(t) = B \cos(\omega t) + C \sin(\omega t) + \frac{A}{1 - (\Omega/\omega)^2} \sin(\Omega t).$$

Applying the initial condition  $x(0) = 0$  yields  $B = 0$ ; applying  $\dot{x}(0) = 0$  gives the final result

$$x(t) = \frac{A}{1 - (\Omega/\omega)^2} \left[ \sin(\Omega t) - \left(\frac{\Omega}{\omega}\right) \sin(\omega t) \right].$$

Note how the amplitude of the oscillation grows as  $\Omega \rightarrow \omega$ . We can ensure the mug's integrity by choosing the amplitude  $A$  such that  $x(t) > -H$ . Since  $|\sin(\Omega t) - (\Omega/\omega) \sin(\omega t)| < (1 + \Omega/\omega)$ , this may be achieved by choosing

$$A < H \frac{1 - (\Omega/\omega)^2}{1 + (\Omega/\omega)} = H \left( 1 - \left(\frac{\Omega}{\omega}\right) \right).$$

This shows that if we're close to resonance,  $\Omega \approx \omega$ , we can only apply a very small forcing amplitude  $A$  if we don't want the mug to hit the floor.

- (c) i. If  $\Omega = \omega$ , the RHS  $r(t) = \omega^2 A \sin(\Omega t)$  is a solution of the homogeneous ODE, therefore we have to multiply the naive ansatz for  $x_P$ , used in the previous question, by the smallest integer power of the independent variable  $t$  for which it ceases to be a solution of the homogeneous ODE. Multiplication by  $t$  does the trick, so we choose

$$x_P(t) = E t \sin(\omega t).$$

As observed on a previous example sheet, trigonometric RHSs are best done by performing the calculation in complex variables. Therefore, we re-write the ODE (2) as

$$\ddot{x} + \omega^2 x = A \omega^2 e^{i\omega t},$$

and only use the imaginary part of the solution. Differentiating the "complexified" ansatz

$$x_P(t) = E t e^{i\omega t}$$

yields

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Insert into the ODE

$$E e^{i\omega t} (2i\omega - \omega^2 t + \omega^2 t) = A \omega^2 e^{i\omega t} \implies E = \frac{A \omega^2}{2i\omega} = -i \frac{A\omega}{2}.$$

Thus

$$x_P(t) = -i \frac{A\omega}{2} t e^{i\omega t} = -i \frac{A\omega}{2} t (\cos(\omega t) + i \sin(\omega t)).$$

Extracting the imaginary part of this complex solution and adding the (unchanged) solution of the homogeneous ODE yields the general solution

$$x(t) = B \cos(\omega t) + C \sin(\omega t) - \frac{A\omega}{2} t \cos(\omega t)$$

for arbitrary constants  $B$  and  $C$ . Applying the initial conditions yields

$$x(t) = \frac{A}{2}(\sin(\omega t) - \omega t \cos(\omega t)).$$

The mug shatters when  $x(t) = -H$ , i.e.

$$-H = \frac{A}{2}(\sin(\omega t_{\text{shatter}}) - \omega t_{\text{shatter}} \cos(\omega t_{\text{shatter}})). \quad (3)$$

- ii. Equation (3) cannot be solved in closed form but zooming in on a plot of the height of mug above the concrete floor versus time,  $x(t) + H$ , for the parameter values specified in the question shows that the impact occurs at  $t_{\text{shatter}} \approx 95.67$  sec.

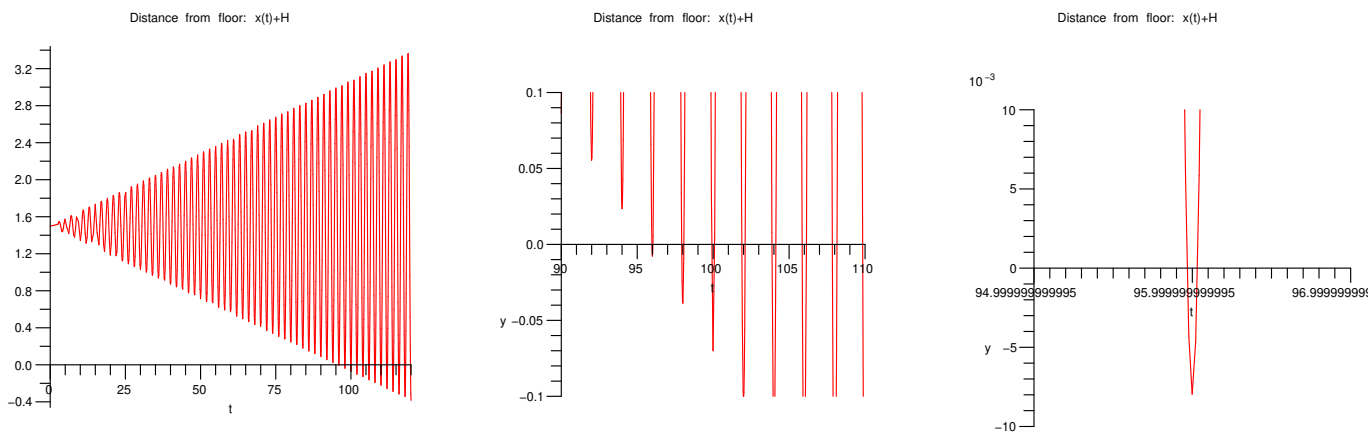


Figure 2: Zooming into a plot of the mug's height above floor,  $x(t) + H$ , shows that the impact occurs at  $t_{\text{shatter}} \approx 95.67$  sec for  $A = 0.01\text{m}$ ,  $H = 1.5\text{m}$ ,  $\omega = \pi \text{sec}^{-1}$

- iii. We can exploit the fact that  $H \gg A$  to obtain an approximation for  $t_{\text{shatter}}$  by assessing the size of the various terms in (3). Since the first term in (3) is bounded,  $|A/2 \sin(\omega t)| < A/2$ , the sum of the two terms on the RHS of (3) will be dominated by the second term when the mug hits the floor. Hence we have

$$H \approx \frac{A\omega}{2} t_{\text{shatter}} \cos(\omega t_{\text{shatter}})$$

Since the amplitude of the oscillation doesn't change much during each period, the mug is likely to hit the floor close to the instant at which  $\cos(\omega t)$  has its maximum value so

$$H \approx \frac{A\omega}{2} t_{\text{shatter}},$$

which yields

$$t_{\text{shatter}} \approx \frac{2H}{\omega A} = 95.49 \text{ sec},$$

accurate to within an error of 0.5%.

## 2. Applications of second-order ODEs: The “Dead Cat Bounce”

- (a) Fig. 3(a) shows the initial conditions, at the moment of impact: The cat’s position is

$$x(t = 0) = 0 \tag{4}$$

and its initial velocity is

$$\left. \frac{dx}{dt} \right|_{t=0} = v_0. \tag{5}$$

Fig. 3(b) illustrates the forces acting on the cat at an instant when its impact has moved the floor downwards by  $x(t)$ : The elastic spring generates an upwards force of magnitude  $c x(t)$ , the damper resists the downward motion with an upward force of magnitude  $k dx/dt$ , and the cat is subject to the (downward) gravitational force  $m g$ . Newton’s law therefore takes the form

$$m \frac{d^2x}{dt^2} = m g - k \frac{dx}{dt} - c x. \tag{6}$$

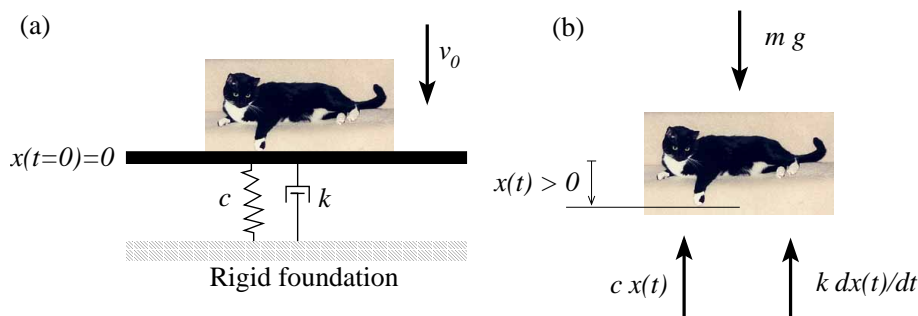


Figure 3: Force balance on the dead cat.

- (b) The cat is *not* glued to the floor, so the floor is not able to “pull” the cat downwards. Hence, when the sum of the forces that the floor exerts on the cat,

$$F_{floor} = k \frac{dx}{dt} + c x,$$

becomes negative (i.e. if the force has a downward direction), the cat will lift off the floor and perform a bounce. During the bounce the cat’s motion is governed by the ODE

$$\frac{d^2x}{dt^2} = g,$$

and this ODE describes the cat’s motion until it hits the ground again, etc.

- (c) Since we’re only trying to find out *if* the cat bounces, we follow its motion until the first bounce (if a bounce occurs!). First, we establish that the ODE is valid at the moment of impact, at  $t = 0$ . The cat’s initial position is  $x(t = 0) = 0$  and its initial velocity is  $dx/dt|_{t=0} = v_0 > 0$ , so at the moment of impact we have

$$F_{floor}(t = 0) = k v_0 \geq 0,$$

showing that the ODE (6) is valid at  $t = 0$ . Does it remain valid for at least a short while afterwards? To examine this, we use the Taylor expansion of  $F_{floor}(t)$  for small times:

$$F_{floor}(t = \Delta t) = \underbrace{F_{floor}|_{t=0}}_{k v_0} + \left. \frac{dF_{floor}}{dt} \right|_{t=0} \Delta t + \mathcal{O}(\Delta t^2) \quad \text{for small } \Delta t, \quad (7)$$

where

$$\left. \frac{dF_{floor}}{dt} \right|_{t=0} = \left[ \frac{d}{dt} \left( k \frac{dx}{dt} + cx \right) \right] \Big|_{t=0} = k \left. \frac{d^2x}{dt^2} \right|_{t=0} + \underbrace{c \left. \frac{dx}{dt} \right|_{t=0}}_{c v_0}.$$

But what is  $d^2x/dt^2|_{t=0}$ ? Since the ODE (6) is valid at  $t = 0$ , we can solve it for  $d^2x/dt^2$ :

$$\left. \frac{d^2x}{dt^2} \right|_{t=0} = g - \underbrace{\frac{k}{m} \left. \frac{dx}{dt} \right|_{t=0}}_{\frac{k}{m} v_0} - \underbrace{\frac{c}{m} x \Big|_{t=0}}_0,$$

so

$$F_{floor}(t = \Delta t) = k v_0 + \left[ k g + v_0 \left( c - \frac{k^2}{m} \right) \right] \Delta t + \mathcal{O}(\Delta t^2).$$

This shows that for  $k > 0$ , the floor initially exerts an upward force onto the cat. Depending on the sign of the term in the square brackets this force either increases or decreases. However, even if the term is very large and negative, we can always choose a sufficiently small value of  $\Delta t$  to ensure that  $F_{floor} \geq 0$  for  $0 < t < \Delta t$ , showing that the ODE remains valid for a short while after the cat's initial impact.

If  $k = 0$  (no damping), we have  $F_{floor}(t = 0) = 0$  and  $dF_{floor}/dt|_{t=0} > 0$ , so the force is initially zero and then increases. Hence  $F_{floor}$  again remains non-negative, at least for a short period after the cat's initial impact.

**Common sense comment:** This should all make perfect sense when you think about the mechanics of the problem: At  $t = 0$  the cat has a finite downward velocity. Following the impact, the cat is being decelerated but its velocity is not going to reverse instantly. Hence the cat will continue to move downwards (creating an upward force from the damper) and thus depress the spring (creating an upward force from the spring). Hence, the resultant force  $F_{floor}$  acts upwards too and the ODE is valid.

(d) We re-write the ODE in its standard form

$$\frac{d^2x}{dt^2} + 2\delta \frac{dx}{dt} + \omega^2 x = g, \quad (8)$$

where

$$\delta = \frac{k}{2m} \quad \text{and} \quad \omega^2 = \frac{c}{m}.$$

Assuming that  $\delta^2 < \omega^2$ , the homogeneous solution is given by

$$x_H(t) = e^{-\delta t} (A \cos(\Omega t) + B \sin(\Omega t))$$



where we have used the shorthand

$$\Omega = \sqrt{\omega^2 - \delta^2}.$$

A particular solution for the ODE (8) is given by

$$x_P = \frac{g}{\omega^2} = \frac{mg}{c},$$

which represents the cat's ultimate equilibrium position (i.e. the position it ends up in when all the bouncing is done...). In this configuration, the cat's weight (a downward force of magnitude  $mg$ ) is balanced by the spring force (an upward force of magnitude  $x_P c$ ).

Applying the initial conditions (4) and (5) to  $x(t) = x_P(t) + x_H(t)$  yields

$$A = -\frac{g}{\omega^2}$$

and

$$B = \frac{v_0 - \delta g/\omega^2}{\Omega}.$$

Here is a particularly neat way of writing the solution:

$$x(t) = \frac{g}{\omega^2} \left( 1 + e^{-\delta t} \left( -\cos(\Omega t) + \frac{v_0 \omega/g - \delta/\omega}{\Omega/\omega} \sin(\Omega t) \right) \right) \quad (9)$$

- (e) Figs. 4 and 5 show plots of  $x(t)$  and  $F_{floor}(t)$  for  $v_0 = 10$  m/sec,  $g = 9.81$  m/sec<sup>2</sup>,  $\omega = \pi$  sec<sup>-1</sup> and  $\delta = 0, 1, 2, 3$  sec<sup>-1</sup>. An increase in damping increases the rate at which the oscillations decay. For sufficiently large  $\delta$ ,  $F_{floor}$  remains positive for all  $t$  and the cat doesn't bounce.

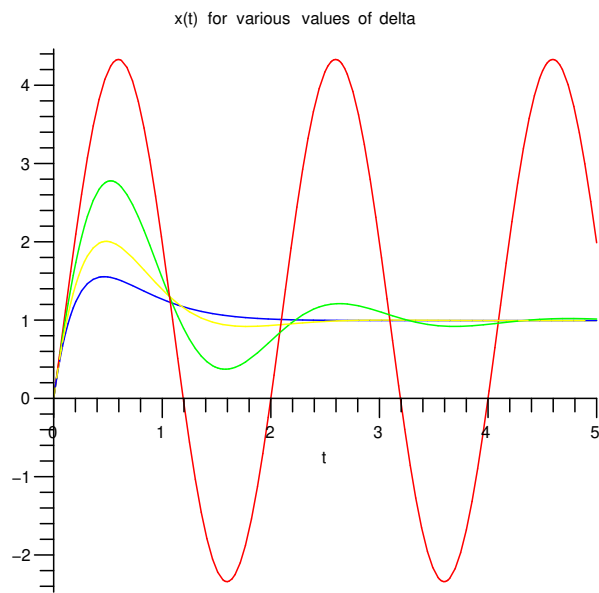


Figure 4: The cat's trajectory for  $v_0 = 10$  m/sec,  $g = 9.81$  m/sec<sup>2</sup>,  $\omega = \pi$  sec<sup>-1</sup> and  $\delta = 0, 1, 2, 3$  sec<sup>-1</sup>. The solution is only valid until  $F_{floor}(t) < 0$  for the first time; see Fig. 5.

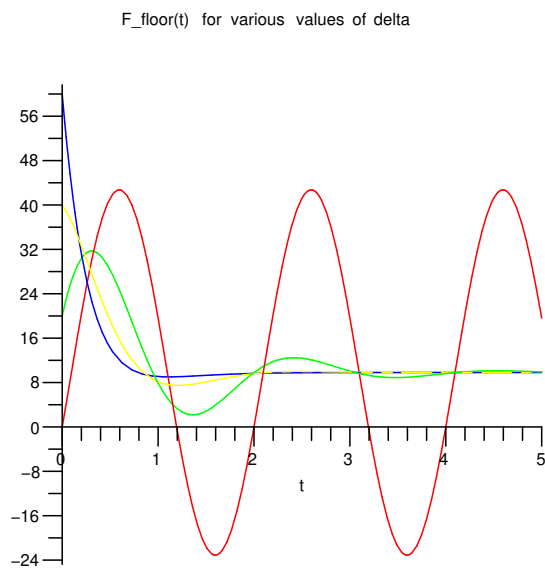


Figure 5: The force exerted by the floor onto the cat,  $F_{floor}(t)$  for  $v_0 = 10$  m/sec,  $g = 9.81$  m/sec<sup>2</sup>,  $\omega = \pi$  sec<sup>-1</sup> and  $\delta = 0, 1, 2, 3$  sec<sup>-1</sup>. The solution is only valid until  $F_{floor}(t) < 0$  for the first time.

(f) If there's no damping ( $\delta = 0$ , which implies  $\Omega = \omega$ ), equation (9) simplifies to

$$x(t) = \frac{g}{\omega^2} \left( 1 + \frac{v_0 \omega}{g} \sin(\omega t) - \cos(\omega t) \right)$$

and

$$F_{floor} = c x,$$

so

$$\frac{F_{floor}}{mg} = 1 - \cos(\omega t) + \frac{v_0 \omega}{g} \sin(\omega t),$$

where we have used the fact that  $\omega^2 = c/m$ . This function changes sign when the two curves  $1 - \cos(\omega t)$  and  $-\Lambda \sin(\omega t)$  (where  $\Lambda = \frac{v_0 \omega}{g} > 0$ ) intersect. Sketching these curves (see Fig. 6) shows that this occurs (regardless of the value of  $\Lambda > 0$ ) in the range  $\pi/\omega < t < 2\pi/\omega$ , so the cat will definitely bounce.

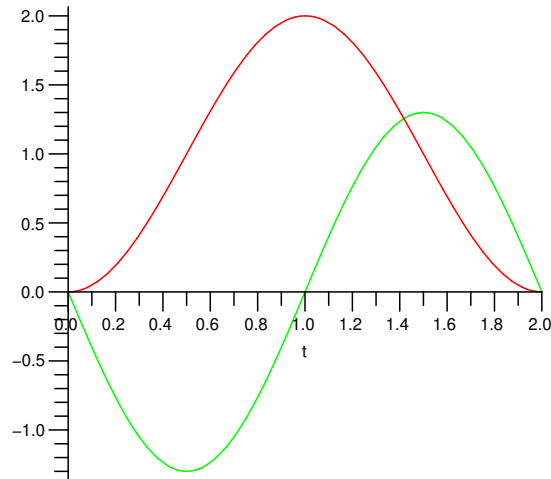


Figure 6: In the absence of damping ( $\delta = 0$ ), the intersection between the curves  $1 - \cos(\omega t)$  and  $-\Lambda \sin(\omega t)$  determines the instant at which  $F_{floor}$  changes sign. The intersection always exists so the cat always bounces. (Sketch for  $\omega = \pi$  and  $\Lambda = 1.3$ ).

### 3. Motivating the use of scaling arguments to simplify ODEs

We know that, following the decay of the transient solution, the solution of

$$m \frac{d^2x}{dt^2} + k \frac{dx}{dt} + cx = F \cos(\Omega t) \quad (10)$$

is given by

$$x_P(t) = A \cos(\Omega t) + B \sin(\Omega t).$$

We wish to determine an approximate solution for  $x_P(t)$  that is appropriate in the limit of “large”  $\Omega$ . Let’s examine the relative sizes of the various terms in (10):

$$\frac{dx_P(t)}{dt} = \Omega(-A \sin(\Omega t) + B \cos(\Omega t))$$

and

$$\frac{d^2x_P(t)}{dt^2} = -\Omega^2(A \cos(\Omega t) + B \sin(\Omega t)).$$

As  $\Omega$  grows, the “inertial term”  $m d^2x/dt^2$  is therefore likely to dominate the other two terms on the LHS of (10), suggesting that the simplified ODE

$$m \frac{d^2x}{dt^2} = F \cos(\Omega t) \quad (11)$$

should provide a good approximation of the system’s behaviour for large  $\Omega$ . The simplified ODE (11) can be integrated twice, yielding

$$x_P(t) = -\frac{F}{m\Omega^2} \cos(\Omega t).$$

The exact solution is given by

$$x_P(t) = A \cos(\Omega t) + B \sin(\Omega t),$$

where

$$A = F \frac{c - m\Omega^2}{(k\Omega)^2 + (c - m\Omega^2)^2} = \frac{F}{m\Omega^2} \frac{\frac{c}{m\Omega^2} - 1}{\left(\frac{k}{m\Omega}\right)^2 + \left(\frac{c}{m\Omega^2} - 1\right)^2}$$

and

$$B = F \frac{k\Omega}{(k\Omega)^2 + (c - m\Omega^2)^2} = \frac{F}{m\Omega^2} \frac{\frac{k}{m\Omega}}{\left(\frac{k}{m\Omega}\right)^2 + \left(\frac{c}{m\Omega^2} - 1\right)^2}.$$

Hence

$$x_P(t) = \frac{F}{m\Omega^2} \left( -\cos(\Omega t) + \mathcal{O}\left(\frac{1}{\Omega}\right) \right) \quad \text{as } \Omega \rightarrow \infty,$$

in agreement with the solution obtained from the simplified ODE.