MATH10222: SOLUTIONS [1] IV

1. Inhomogeneous linear second-order ODEs with constant coefficients

(a) Exploiting linearity

i.

$$\ddot{y} + 3\,\dot{y} + 2\,y = 4\,e^{2\,t} \quad (I)$$

- Corresponding homogeneous equation:

$$\ddot{y} + 3\,\dot{y} + 2\,y = 0 \quad (H)$$

- Characteristic equation:

$$\lambda^2 + 3\,\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = -1, -2 \quad \text{distinct, real roots.}$$

- So the general solution of (H) is

$$y_H(t) = C\,e^{-t} + D\,e^{-2\,t}.$$

- Particular solution: In (I) $r(t) = 4\,e^{2\,t}$, so try $y_P = A\,e^{2\,t}$, $\dot{y}_P = 2\,A\,e^{2\,t}$, $\ddot{y}_P = 4\,A\,e^{2\,t}$. Substitute and solve for A ($e^{2\,t}$ will cancel through):

$$4\,A + 3\,(2\,A) + 2\,A = 4 \quad \Longleftrightarrow \quad A = \frac{1}{3}.$$

  So

$$y_P = \frac{1}{3}\,e^{2\,t}.$$

- Hence the general solution of (I) is

$$y = y_H + y_P = C\,e^{-t} + D\,e^{-2\,t} + \frac{1}{3}\,e^{2\,t}.$$

  for arbitrary constants C and D.

ii.

$$\ddot{y} + 3\,\dot{y} + 2\,y = 7 \quad (I)$$

- The homogeneous equation is the same as in the previous example, so

$$y_H(t) = C\,e^{-t} + D\,e^{-2\,t}$$

  as before.
- Now we need a particular solution for the constant RHS $r(t) = 7$. Try $y_P = A$, substitute into the ODE, and solve for A.

$$A = \frac{7}{2}.$$

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- Hence the general solution is

$$y = y_H + y_P = C\,e^{-t} + D\,e^{-2\,t} + \frac{7}{2}.$$

iii.

$$\ddot{y} + 3\,\dot{y} + 2\,y = 4\,e^{2\,t} + 7.$$

The homogeneous equation is the same, yet again, and the RHS is the sum of the two RHSs considered in the previous examples. Because the ODE is linear, we can simply add the the particular solutions obtained in these cases. Therefore, the general solution is given by

$$y = C\,e^{-t} + D\,e^{-2\,t} + \frac{7}{2} + \frac{1}{3}\,e^{2\,t}.$$

Oh, aren't linear equations beautiful (or boring?).

(b) **Using complex variables for trigonometric RHSs**

We're looking for the general solutions of

$$\ddot{y} + 2\,\dot{y} + 2\,y = \begin{pmatrix} 10\,\cos t \\ 10\,\sin t \end{pmatrix} \quad (I)$$

- First consider the homogeneous equation

$$\ddot{y} + 2\,\dot{y} + 2\,y = 0 \quad (H)$$

which is the same in both cases.

- Characteristic equation:

$$\lambda^2 + 2\,\lambda + 2 = 0 \Longrightarrow \lambda_{1,2} = -1 \pm i,$$

i.e. complex conjugate roots.

- Thus, for both cases the general solution of the homogeneous ODE is given by

$$y_H(t) = e^{-t}\,(C\,\cos t + D\,\sin t).$$

- To obtain the particular solution for both cases simultaneously, we exploit the fact that $\cos t = \mathrm{Re}(e^{it})$ and $\sin t = \mathrm{Im}(e^{it})$ and determine a (complex) particular solution for

$$\ddot{y} + 2\,\dot{y} + 2\,y = 10\,e^{it}, \quad (C)$$

and then extract the real and imaginary parts.

- Given that $r(t) = e^{it}$, we try $y_p = Ae^{it}$, $\dot{y}_P = i\,A\,e^{i\,t}$, $\ddot{y}_P = -A\,e^{i\,t}$.
- Substitute into (C) and cancel the common factor $e^{i\,t}$:

$$-A + 2\,(i\,A) + 2\,A = 10$$

$$A\,(1 + 2\,i) = 10$$

$$A = 2\,(1 - 2\,i)$$

- So
$$y_P = 2\,(1-2\,i)\,e^{i\,t} = (2-4\,i)\,(\cos t + i\,\sin t).$$
- Now extract the real and imaginary parts:
$$\mathrm{Re}(y_P) = 2\,\cos t + 4\,\sin t \quad \text{and} \quad \mathrm{Im}(y_P) = -4\,\cos t + 2\,\sin t.$$
- So the general solution of (I) is
$$y = e^{-t}\,(C\,\cos t + D\,\sin t) + \left(\begin{array}{c} 2\,\cos t + 4\,\sin t \\ -4\,\cos t + 2\,\sin t \end{array}\right) \quad (I)$$

(c) **Degenerate and non-degenerate cases for RHSs of exponential form**

i.
$$\ddot{y} + 3\,\dot{y} + 2\,y = 2\,e^{-t}$$

- Characteristic equation: $\lambda^2 + 3\,\lambda + 2 = 0$ *i.e.* $\lambda_{1,2} = -1,\ -2$, so
$$y_H(t) = C\,e^{-t} + D\,e^{-2\,t}.$$
- The RHS $r(t) = 2\,e^{-t}$, has the same form as one of the fundamental solutions, so an ansatz of the form $y_p \sim e^{-t}$ won't work (Try it!). Use
$$y_P = A\,t\,e^{-t}, \qquad \dot{y}_P = (A - A\,t)\,e^{-t}, \quad \ddot{y}_P = (-2\,A + A\,t)\,e^{-t}.$$
instead.
- Substitute into the ODE and cancel the common factor e^{-t}:
$$\begin{aligned} -2\,A + A\,t + 3\,A - 3\,A\,t + 2\,A\,t &= 2 \\ t\,(A - 3\,A + 2\,A) + A &= 2 \qquad \text{so} \quad A = 2. \end{aligned}$$
- The general solution is
$$y = C\,e^{-t} + D\,e^{-2\,t} + 2\,t\,e^{-t}$$
for arbitrary constants C and D.

ii.
$$\ddot{y} + 4\,\dot{y} + 4\,y = e^{-2\,t}$$

- Characteristic equation: $\lambda^2 + 4\,\lambda + 4 = 0$ *i.e.* $\lambda_{1,2} = -2$, so
$$y_H(t) = (C + D\,t)\,e^{-2\,t}.$$
- The RHS $r(t) = e^{-2\,t}$, has the same form as the first fundamental solution and $te^{-2\,t}$ has the same form as the second one, so try
$$y_P = A\,t^2\,e^{-2\,t}, \quad \dot{y}_P = (2\,A\,t - 2\,A\,t^2)\,e^{-2\,t}, \quad \ddot{y}_P = (-8\,A\,t + 4\,A\,t^2 + 2\,A)\,e^{-2\,t}$$
instead.

- Substitute into the ODE and cancel the common factor e^{-2t}:

$$\begin{aligned} -8\,A\,t + 4\,A\,t^2 + 2\,A + 8\,A\,t - 8\,A\,t^2 + 4\,A\,t^2 &= 1 \\ t^2\,(4\,A - 8\,A + 4\,A) + t\,(-8\,A + 8\,A) + 2\,A &= 1 \\ A &= \frac{1}{2}. \end{aligned}$$

- The general solution is

$$y = (C + D\,t + \frac{t^2}{2})\,e^{-2\,t}.$$

for arbitrary constants C and D.

iii.

$$\ddot{y} + 2\,\dot{y} + 2\,y = 5\,\cosh t$$

- The homogeneous equation, $\ddot{y} + 2\,\dot{y} + 2\,y = 0$ is the same as in part 1b, so

$$y_H(t) = e^{-t}\,(C\,\cos t + D\,\sin t),$$

as before.
- Given the RHS $r(t) = 5\,\cosh t$ we'd like to try an ansatz of the form $y_p(t) = A\,\cosh t$. When differentiated, this will also produce terms that are proportional to $\sinh t$. To balance these, we try $y_p(t) = A\,\cosh t + B\,\sinh t$:

$$y_p = A\,\cosh t + B\,\sinh t,$$

$$\dot{y}_P = A\,\sinh t + B\,\cosh t,$$

$$\ddot{y}_P = A\,\cosh t + B\,\sinh t = y_p.$$

- Substitute into the ODE:

$$\begin{aligned} 3\big(A\,\cosh t + B\,\sinh t\big) + 2\big(A\,\sinh t + B\,\cosh t\big) &= 5\,\cosh t. \\ \cosh t\,\big(3A + 2B - 5\big) + \sinh t\,\big(3B + 2A\big) = 0 \end{aligned}$$

- So $3A + 2B - 5 = 0$ and $3B + 2A = 0 \implies A = 3,\ B = -2$.
- The general solution is therefore

$$y = e^{-t}\,(C\,\cos t + D\,\sin t) + 3\,\cosh t - 2\sinh t$$

for arbitrary constants C and D.

iv.

$$\ddot{y} + 3\,\dot{y} + 2\,y = 2\,\cosh t$$

- The homogeneous equation, $\ddot{y} + 3\,\dot{y} + 2\,y = 0$ is the same as in part 1(c)i, so

$$y_H(t) = C\,e^{-t} + D\,e^{-2\,t}$$

as before.

- As in the previous problem, the RHS $r(t) = 2\cosh t$ suggests trying a particular solution that contains sinh and cosh terms. However, $r(t) = 2\cosh t$ contains one of the two fundamental solutions as $2\cosh t = e^t + e^{-t}$. Therefore we must look for a particular integral of the form

$$y_P = A\,t\,e^{-t} + B\,e^t, \quad \dot{y}_P = e^{-t}\,(A - A\,t) + B\,e^t, \quad \ddot{y}_P = e^{-t}\,(-2\,A + A\,t) + B\,e^t.$$

- Substitute into the ODE:

$$\begin{aligned} e^{-t}\,(-2\,A + A\,t + 3\,A - 3\,A\,t + 2\,A\,t) + e^t\,(B + 3\,B + 2\,B) &= e^t + e^{-t} \\ A\,e^{-t} + 6\,B\,e^t &= e^t + e^{-t} \\ A = 1, \qquad B &= \frac{1}{6}. \end{aligned}$$

- The general solution is therefore

$$y = C\,e^{-t} + D\,e^{-2\,t} + t\,e^{-t} + \frac{1}{6}\,e^t$$

for arbitrary constants C and D.

(d) **Degenerate and non-degenerate cases for polynomial RHSs**

i.

$$\ddot{y} + 3\,\dot{y} + 2\,y = 1 + t^2$$

- The homogeneous ODE is the same as in question 1(a)i so

$$y_H(t) = D\,e^{-t} + E\,e^{-2\,t}.$$

- Particular solution: The RHS $r(t) = 1 + t^2$ suggests using a complete second-order polynomial as an ansatz for y_P:

$$y_P = A + B\,t + C\,t^2$$

(The term that's linear in t is needed to balance the term that arises from the differentiation of the t^2-term. Try omitting it if you don't believe it!)

$$\dot{y_P} = B + 2\,C\,t,$$
$$\ddot{y_P} = 2\,C.$$

- Insert into the ODE:

$$2\,C \; + 3\big(B + 2\,C\,t\big) + 2\big(A + B\,t + C\,t^2\big) = 1 + t^2.$$

- Collect powers of t:

$$\big(2\,C + 3B + 2\,A - 1\big) + \big(6\,C + 2\,B\big)\,t + \big(2\,C - 1\big)\,t^2 = 0.$$

- Setting the coefficients to zero yields:

$$C = 1/2, \quad B = -3/2, \quad A = 9/4.$$

- Hence the general solution is

$$y = y_P + y_H = D\,e^{-t} + E\,e^{-2\,t} + \frac{9}{4} - \frac{3}{2}\,t + \frac{1}{2}\,t^2.$$

ii.

$$\ddot{y} + 2\,\dot{y} = 1 + t^2$$

- The homogeneous solution is

$$y_H(t) = D\, + E\,e^{-2\,t}.$$

- Particular solution: As in the previous case, the RHS $r(t) = 1 + t^2$ suggests using a complete second-order polynomial as an ansatz for the particular solution:

$$\widehat{y_P} = A + B\,t + C\,t^2.$$

However, here this isn't going to work (try it!) because the constant term is a solution of the homogeneous ODE $\Longrightarrow$ multiply the ansatz $\widehat{y_P}$ by t^m where m is the smallest positive integer for which none of the terms in $t^m\,\widehat{y_P}$ are solutions of the homogeneous ODE. In our example, $m = 1$ does the trick, so we choose

$$\begin{aligned} y_P &= A\,t + B\,t^2 + C\,t^3, \\ \dot{y_P} &= A\, + 2\,B\,t + 3\,C\,t^2, \\ \ddot{y_P} &= 2\,B + 6\,C\,t. \end{aligned}$$

- Insert into the ODE:

$$\big(2\,B + 6\,C\,t\big) + 2\big(A\, + 2\,B\,t + 3\,C\,t^2\big) = 1 + t^2.$$

- Collect powers of t:

$$\big(2\,B + 2\,A - 1\big) + \big(6\,C + 4\,B\big)\,t + \big(6\,C - 1\big)\,t^2 = 0.$$

- Setting the coefficients to zero yields:

$$C = 1/6, \quad B = -1/4, \quad A = 3/4.$$

- Hence the general solution is

$$y = y_P + y_H = D + E\,e^{-2\,t} + \frac{3}{4}\,t - \frac{1}{4}\,t^2 + \frac{1}{6}\,t^3.$$

iii.

$$\ddot{y} = 1 + t^2$$

- The homogeneous solution is

$$y_H(t) = D\, + E\,t.$$

- Particular solution: Here, using a complete second-order polynomial as an ansatz for y_P won't work because the constant *and* linear terms are solutions of the homogeneous equation. We have to multiply the naive ansatz $y_P^{[naive]} = A + B\,t + C\,t^2$ by t^2 to ensure that none of its terms are solutions of the homogeneous ODE:

$$\begin{aligned} y_P &= A\,t^2 + B\,t^3 + C\,t^4, \\ \dot{y_P} &= 2\,A\,t + 3\,B\,t^2 + 4\,C\,t^3, \\ \ddot{y_P} &= 2\,A + 6\,B\,t + 12\,C\,t^2. \end{aligned}$$

- Insert into the ODE:

$$\left(2\,A + 6\,B\,t + 12\,C\,t^2\right) = 1 + t^2$$

- Collect powers of t:

$$\left(2\,A - 1\right) + \left(6\,B\right)t + \left(12\,C - 1\right)t^2 = 0.$$

- Setting the coefficients to zero yields:

$$C = 1/12, \quad B = 0, \quad A = 1/2.$$

- Hence the general solution is

$$y = y_P + y_H = D + E\,t + \frac{1}{2}\,t^2 + \frac{1}{12}\,t^4.$$

[Of course, we could have found the solution directly by integrating the ODE twice!]

2. **Linear ODEs with non-constant coefficients: Euler's ODE**

(a) Given that your lecturer has kindly provided you with an ansatz, it is quite legitimate for you to insert it into the ODE and stare at it for a while. After a while you should suddenly notice (with immense delight!) that a common factor t^n can be extracted from all terms so that, just as in the case of an $e^{\lambda t}$-ansatz for a constant-coefficient ODE, we are left with an (algebraic) equation that determines the values of n:

$$\text{Substitute} \quad y = t^n \quad \text{into} \quad a\,t^2\,\ddot{y}(t) + b\,t\,\dot{y}(t) + c\,y(t) = 0:$$

$$\begin{aligned} a\,t^2\,n\,(n-1)\,t^{n-2} + b\,t\,n\,t^{n-1} + c\,t^n &= 0, \\ a\,n\,(n-1) + b\,n + c &= 0 \\ a\,n^2 + (b-a)\,n + c &= 0. \end{aligned}$$

This is another "characteristic polynomial".

Is there any theory behind this? Yes, and it's useful to know about it, so keep reading. We start with the seemingly trivial observation that the ODE has to be satisfied for *all* values of the independent variable $t \in I$:

$$a\,t^2\,\ddot{y}(t) + b\,t\,\dot{y}(t) + c\,y(t) = 0 \qquad \forall x \in I.$$

This requires the sum of the three terms, $a\,t^2\,\ddot{y}(t)$, $b\,t\,\dot{y}(t)$ and $c\,y(t)$ to vanish for all values of $t \in I$. There are two particularly "easy" ways to achieve this:

- We make the three terms vanish individually. If $c \neq 0$, this requires $y \equiv 0$ and therefore only yields the trivial solution. Not a particularly clever idea then...
- We ensure that the three terms have the same functional dependence on the independent variable so that:

$$a \underbrace{t^2\,\ddot{y}(t)}_{\mathcal{A}\mathcal{F}(t)} + b \underbrace{t\,\dot{y}(t)}_{\mathcal{B}\mathcal{F}(t)} + c \underbrace{y(t)}_{\mathcal{C}\mathcal{F}(t)} = 0, \qquad (1)$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are constants. If (!) this can be achieved, we can re-write the ODE as

$$\mathcal{F}(t)\big(a\,\mathcal{A} + b\,\mathcal{B} + c\,\mathcal{C}\big) = 0,$$

and obtain the coefficients from the algebraic condition

$$a\,\mathcal{A} + b\,\mathcal{B} + c\,\mathcal{C} = 0.$$

Under what conditions will this work? The middle term in (1) requires that

$$t\,\frac{dy}{dt} = \mathcal{B}\,\mathcal{F}(t), \qquad (2)$$

while the last term provides an expression for dy/dt:

$$y(t) = \mathcal{C}\,\mathcal{F}(t) \quad \Longrightarrow \quad \frac{dy}{dt} = \mathcal{C}\,\frac{d\mathcal{F}(t)}{dt}. \qquad (3)$$

Combining (2) and (3) shows that

$$t\,\frac{d\mathcal{F}(t)}{dt} = n\,\mathcal{F}(t) \qquad (4)$$

where $n = \mathcal{B}/\mathcal{C}$. Equation (4) can be solved by separation of variables which shows that $\mathcal{F}(t)$ must have the form

$$\mathcal{F}(t) = A\,t^n.$$

The same result is obtained by equating the first and second terms in (1). Nice, innit?

(b) Let's try this for

$$t^2\,\ddot{y} + 2\,t\,\dot{y} - 2\,y = 0.$$

- Inserting $y \sim t^n$ yields

$$t^2\,n(n-1)\,t^{n-2} + 2\,t\,n\,t^{n-1} - 2\,t^n = 0 \quad \Longleftrightarrow \quad t^n\,(n^2 + n - 2) = 0.$$

Now $t \neq 0$ in general, so the possible values of n are $n = 1$ and $n = -2$, giving two solutions

$$y_1 = t \quad \text{and} \quad y_2 = \frac{1}{t^2}.$$

The two solutions are nonzero and linearly independent therefore they constitute a set of fundamental solutions for the linear homogeneous ODE. The general solution is therefore

$$y = A\,t + \frac{B}{t^2}$$

for any constants A and B.

(c) Using the ansatz $y \sim t^n$ in the ODE

$$t^2\,\ddot{y} - t\,\dot{y} + y = 0,$$

yields the characteristic polynomial

$$n\,(n-1) - n + 1 = n^2 - 2\,n + 1 = (n-1)^2 = 0,$$

which has the double root $n_{1,2} = 1$. Hence we only obtain a single solution $y_1(t) = A\,t$.

A second solution may be obtained via the "reduction of order" method, discussed on the previous example sheet, by posing $y_2(t) = g(t)\,y_1(t)$, i.e.

$$\begin{aligned} y_2 &= t\,g(t), \\ \dot{y}_2 &= t\,\dot{g}(t) + g(t), \\ \ddot{y}_2 &= 2\,\dot{g}(t) + t\,\ddot{g}(t). \end{aligned}$$

Substitute into the ODE:

$$\begin{aligned} t^2\left(2\,\dot{g}(t) + t\,\ddot{g}(t)\right) - t\left(t\,\dot{g}(t) + g(t)\right) + t\,g(t) &= 0, \\ t^3\,\ddot{g}(t) + t^2\,\dot{g}(t) &= 0. \end{aligned}$$

This is a first-order ODE for $v = \dot{g}$:

$$\begin{aligned} t\,\dot{v}(t) + v &= 0, \\ t\,\frac{dv}{dt} + v &= 0. \end{aligned}$$

Separate variables

$$\begin{aligned} \int \frac{1}{t}\,dt &= -\int \frac{1}{v}\,dv \\ \ln|t| + D = \ln|t| + \ln|C| = \ln|Ct| &= -\ln|v| = \ln|1/v|, \\ v(t) &= \frac{1}{Ct} = \frac{dg}{dt}. \end{aligned}$$

Separate again:

$$\int dg = g(t) = \int \frac{1}{Ct}\,dt = \frac{1}{C}\ln|t| + D.$$

The constants of integration are irrelevant and a second, linearly independent solution is obtained by choosing, $C = 1$ and $D = 0$, say, yielding $g(t) = \ln|t|$ and thus

$$y_2(t) = t\,\ln|t|.$$

So the general solution of the ODE is given by

$$y(t) = A\,t + B\,t\,\ln|t|.$$

3. **Non-linear ODEs with special properties**

(a)

$$y\,y'' = (y')^2.$$

- This ODE is autonomous so we substitute

$$\frac{dy}{dx} = v \quad \Longrightarrow \quad \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\,\frac{dv}{dy}.$$

- This transforms the ODE into a separable first-order ODE for $v(y)$:

$$y\,v\,\frac{dv}{dy} = v^2.$$

Note: $v = 0 \quad \Longrightarrow \quad y(x) = const.$ is a solution.

$$y\,\frac{dv}{dy} = v.$$

Separate:

$$\int \frac{1}{y}\,dy = \int \frac{1}{v}\,dv,$$

$$\ln|y| + E = \ln|y| + \ln|C| = \ln|Cy| = \ln|v|$$

$$v = C\,y$$

- Now back-substitute

$$v = \frac{dy}{dx} = C\,y,$$

and separate again

$$\int \frac{1}{y}\,dy = \int C\,dx$$

$$\ln|y| = C\,x + D$$

$$y = e^{Cx+D} = A\,e^{Cx}.$$

(b)

$$y'' = \frac{2x^2}{(y')^2}$$

subject to

$$y'(1) = 2^{1/3} \quad \text{and} \quad y(1) = 2^{-2/3}.$$

- The dependent variable, y, does not appear in the ODE $\Longrightarrow$ the ODE is a first-order ODE for $v(x) = dy/dx$:

$$\frac{dv}{dx} = \frac{2x^2}{v^2}.$$

Separate:

$$\int v^2\,dv = \int 2\,x^2 dx,$$

$$\frac{1}{3}\,v^3 = \frac{2}{3}\,(x^3 + C)$$

$$v(x) = \big(2\,(x^3 + C)\big)^{1/3}.$$

Apply initial condition: $y'(1) = v(1) = 2^{1/3} \quad \Longrightarrow \quad C = 0.$

$$v(x) = \frac{dy}{dx} = 2^{1/3}\,x$$

Separate again

$$\int dy = 2^{1/3} \int x\,dx$$

$$y(x) = 2^{1/3}\,\frac{x^2}{2} + D = \frac{x^2}{2^{2/3}} + D$$

Apply initial condition $y(1) = 2^{-2/3}$: $D = 0$, so the solution of the initial value problem is:

$$y(x) = 2^{-2/3}\,x^2.$$