

MATH10222: SOLUTIONS TO EXAMPLE SHEET¹

II

1. Existence, uniqueness and graphical solutions

- (a) To apply the existence and uniqueness theorem, rewrite the ODE in its standard form $y' = f(x, y)$. The existence and uniqueness theorem guarantees the existence of a unique solution in the vicinity of the point (X, Y) if $f(x, y)$ and $\frac{\partial f(x, y)}{\partial y}$ are continuous functions of x and y in the vicinity of (X, Y) .

For our ODE,

$$f(x, y) = \frac{x - 1}{y}$$

and

$$\frac{\partial f(x, y)}{\partial y} = -\frac{x - 1}{y^2},$$

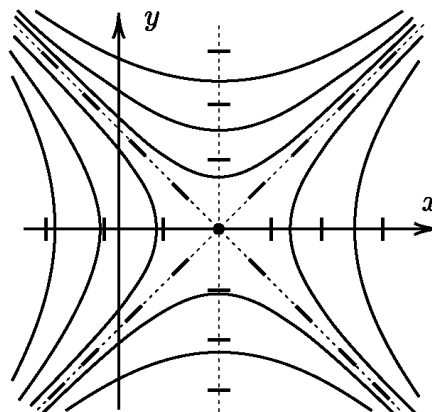
therefore the existence of a unique solution in the vicinity of (X, Y) is guaranteed for all $\{(X, Y) \mid Y \neq 0\}$.

The ODE is nonlinear, therefore the existence and uniqueness theorem *only* ensures the existence in the vicinity of (X, Y) , not for all values of x .

- (b) Isoclines (lines along which the solution of the ODE has the same slope) are given by $y' = (x - 1)/y = c$, a constant. Thus the isocline on which the solution has slope c is given by $y_{iso} = (x - 1)/c$. These are straight lines passing through $(x, y) = (1, 0)$ with slope $1/c$. Here are a few “obvious” ones:

- $y' = 0$ on the vertical line $x = 1$.
- $y' = \infty$ on the horizontal line $y = 0$, i.e. on the x -axis.
- $y' = 1$ on $y = x - 1$
- $y' = -1$ on $y = -(x - 1)$

Here’s a sketch of these isoclines and the corresponding integral curves:



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There's a critical point at $(x, y) = (1, 0)$ where the isoclines intersect.

All solution curves appear to approach the asymptotes $y = \pm(x - 1)$ as $x \rightarrow \pm\infty$.

(c) The ODE is separable:

$$y \frac{dy}{dx} = x - 1,$$

$$\int y \, dy = \int (x - 1) \, dx,$$

$$\frac{1}{2}y^2 = \frac{1}{2}(x - 1)^2 + A \quad \text{for any constant } A,$$

$$y = \pm\sqrt{(x - 1)^2 + C} \quad \text{for any constant } C (= 2A).$$

- (d)
- As $x \rightarrow \pm\infty$, we have $(x - 1)^2 \gg |C|$ for any (finite) value of the constant C so the lines $y = \pm(x - 1)$ are indeed asymptotes for all solutions.
 - For $C = 0$, we obtain two solutions $y = \pm(x - 1)$ – the two asymptotes that emerge from the critical point.
 - If $C > 0$, the solution curves pass through the line $x = 1$ at either $y = \sqrt{C}$ or $y = -\sqrt{C}$, corresponding the solutions above or below the critical point.
 - If $C < 0$ the (real) solutions can't reach $x = 1$ – the solutions intersects the x -axis with infinite slope at $x = 1 \pm \sqrt{-C}$. These correspond to the solution to the right and left of the critical point.

(e) Existence and uniqueness was guaranteed, at least locally, if $Y \neq 0$. The sketch shows what goes wrong if we apply initial conditions on the x -axis: For each initial condition of the form $y(x = X) = 0$, there are two possible solutions – one with $y \geq 0$, the other one with $y \leq 0$.

Regarding the existence of solutions: Recall that for nonlinear ODEs the existence and uniqueness theorem only provides local results: Existence of the solution close to the initial conditions does not ensure its existence for all values of x . In our example, consider the family of solutions that cross the y -axis, i.e. those with initial conditions of the form $y(x = 0) = Y$. While the solutions for $|Y| > 1$ exist for all values of x , those for $|Y| < 1$ only exist over a limited range of x -values, up to the point where they intersect the x -axis.

2. Separable ODEs

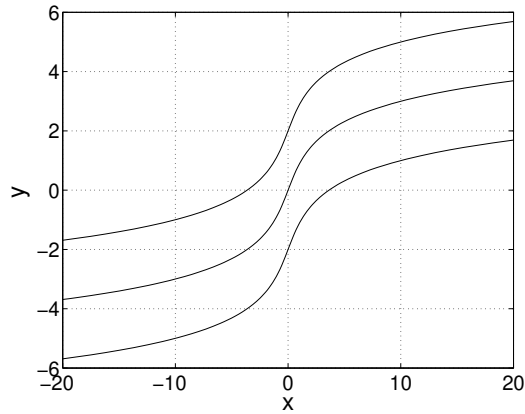
(a)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

Separate and integrate

$$\int dy = y = \int \frac{1}{\sqrt{1 + x^2}} dx + C = \operatorname{arcsinh} x + C.$$

This is the general solution. Here's a plot of the solution for various values of the constant C .



The solution curves all have the same shape. Variations in C shift them along the y -axis.

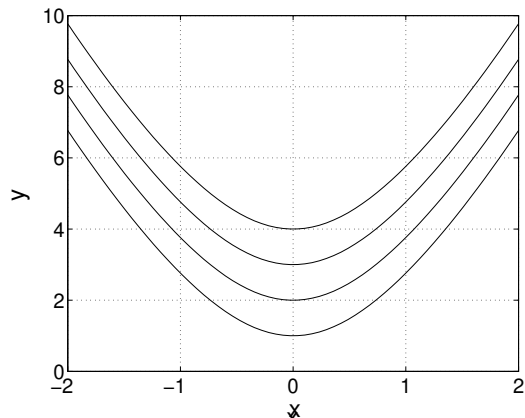
(b)

$$\frac{dy}{dx} = \frac{4x}{(1+x^2)^{1/3}}$$

Separate and integrate, using the substitution $z = 1 + x^2$. This yields

$$y = 3(1+x^2)^{2/3} + C.$$

Here's a sketch of the solutions:



Again, the constant C simply shifts the position of the solution curves.

(c)

$$\frac{dy}{dx} = \frac{-2y}{x-2}$$

Observations: (i) $y \equiv 0$ is a solution. (ii) If $y_1(x)$ is a solution of the ODE then $y_2(x) = -y_1(x)$ is a solution, too.

Separate

$$\frac{1}{y} \frac{dy}{dx} = -\frac{2}{x-2} \quad \text{for } y \neq 0$$

(Note that we've dealt with the case $y = 0$ already: It's also a solution!) and integrate

$$\int \frac{1}{y} dy = - \int \frac{2}{x-2} dx.$$

$$\ln |y| = -2 \ln |x - 2| + C$$

for any constant C . Rewrite

$$\ln |y| = \ln |x - 2|^{-2} + \ln |K|,$$

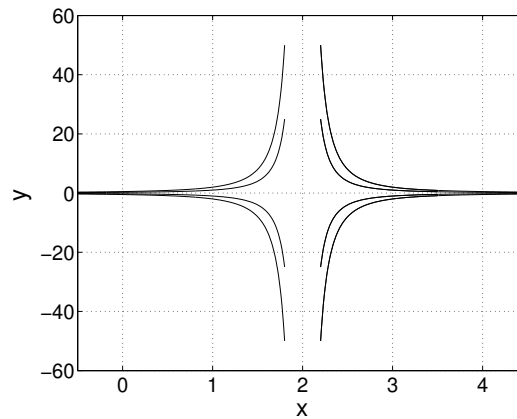
for another constant, K , and combine the logarithms:

$$\ln \left| \frac{y(x-2)^2}{K} \right| = 0 \quad \text{only for } K \neq 0$$

so

$$y = \frac{K}{(x-2)^2} \quad \text{for } K \in \mathbb{R} \text{ since } y \equiv 0 \text{ is a solution too!}$$

The arbitrary constant K **multiplies** the function. If we change K the **shape** of the solution changes.



Note that the solution $y \equiv 0$ is an asymptote for all solutions as $x \rightarrow \pm\infty$.

(d)

$$\sqrt{1+x^2} \frac{dy}{dx} = y$$

Observations: (i) $y \equiv 0$ is a solution. (ii) If $y_1(x)$ is a solution of the ODE then $y_2(x) = -y_1(x)$ is a solution, too.

Separate

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \quad \text{for } y \neq 0,$$

and integrate

$$\int \frac{1}{y} dy = \int \frac{dx}{\sqrt{1+x^2}}$$

$$\ln |y| = \operatorname{arcsinh} x + C$$

Rewrite, using the hint,

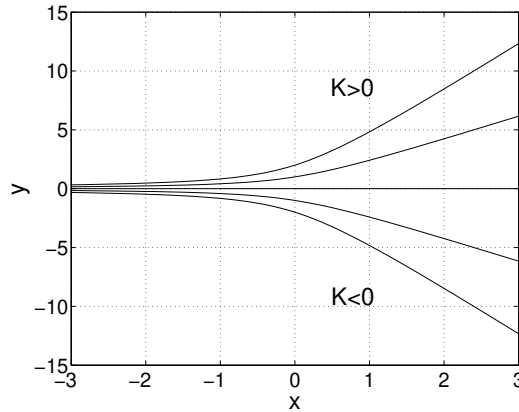
$$\ln |y| = \ln(x + \sqrt{1+x^2}) + \ln |K| = \ln |K(x + \sqrt{1+x^2})|$$

so

$$y = K(x + \sqrt{1+x^2}) \quad \text{for } K \in \mathbb{R}$$

since $y \equiv 0$ is also a solution.

Here is a sketch of the solution



As in the previous example, the constant of integration changes the shape of the solution. The solution $y \equiv 0$ is an asymptote for $x \rightarrow -\infty$.

3. Initial value problems

(a) We have calculated the general solution of the ODE in question 2a:

$$y(x) = \operatorname{arcsinh} x + C$$

Applying the initial condition $y(0) = 5$ yields $5 = \operatorname{arcsinh} 0 + C = C$ so the solution of the initial value problem is

$$y(x) = \operatorname{arcsinh} x + 5.$$

(b) We have calculated the general solution of the ODE in question 2d:

$$y = K(x + \sqrt{1 + x^2})$$

Applying the initial condition $y(0) = -3$ yields $-3 = K(0 + \sqrt{1 + 0}) = K$ so the solution of the initial value problem is

$$y = -3(x + \sqrt{1 + x^2}).$$

4. First-order ODEs of homogeneous type

(a)

$$x y \frac{dy}{dx} + x^2 + y^2 = 0. \tag{1}$$

Assuming that $x \neq 0, y \neq 0$, we rewrite this as

$$\frac{dy}{dx} = -\frac{x}{y} - \frac{y}{x},$$

which shows that the equation is a first-order ODE of homogeneous type.

Put $y(x) = z(x)x$, thus $\frac{dy}{dx} = z + x \frac{dz}{dx}$. The ODE becomes

$$z + x \frac{dz}{dx} = -\frac{1}{z} - z = -\frac{1 + z^2}{z}$$

i.e.

$$x \frac{dz}{dx} = -\frac{1 + 2z^2}{z}.$$

Separate

$$\frac{z}{1 + 2z^2} \frac{dz}{dx} = -\frac{1}{x}$$

$$\int \frac{z}{1 + 2z^2} dz = -\int \frac{1}{x} dx$$

[Use the substitution $u = 1 + 2z^2$]

$$\frac{1}{4} \ln |1 + 2z^2| = -\ln |x| + C$$

$$\ln |1 + 2z^2|^{\frac{1}{4}} = -\ln |x| + \ln |K| = \ln |K/x|$$

$$1 + 2z^2 = \left(\frac{K}{x}\right)^4$$

$$2\frac{y^2}{x^2} = \left(\frac{K}{x}\right)^4 - 1$$

$$y = \pm x \sqrt{\frac{1}{2} \left(\left(\frac{K}{x}\right)^4 - 1 \right)}$$

This is the general solution for $x \neq 0$. Note that for $x = 0$ the coefficient multiplying dy/dx in (1) vanishes – this is always a sign of trouble!

(b)

$$x^2 \frac{dy}{dx} + y^2 - xy = 0$$

Observation: $y \equiv 0$ is a solution.

Rewriting the ODE as

$$\frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2}$$

shows that the equation is a first-order ODE of homogeneous type.

Put $y(x) = z(x)x$, thus $\frac{dy}{dx} = z + x \frac{dz}{dx}$. The ODE becomes

$$z + x \frac{dz}{dx} = z - z^2$$

i.e.

$$x \frac{dz}{dx} = -z^2.$$

Separate,

$$-\frac{1}{z^2} \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{1}{z} = \ln |x| + C$$

$$\frac{1}{y} = \frac{\ln |x| + C}{x} \quad (x \neq 0, y \neq 0)$$

$$y = \frac{x}{\ln|x| + C}$$

This is the general solution for $x \neq 0, y \neq 0$. We know that $y \equiv 0$ is another solution. At $x = 0$ the RHS of the ODE is singular and the solution is not defined.

5. First-order linear ODEs

(a)

$$(1 - x^2) \frac{dy}{dx} - xy = 1 \tag{2}$$

is a linear first-order ODE.

Rearrange into the standard form $dy/dx + p(x)y(x) = q(x)$:

$$\frac{dy}{dx} - \frac{x}{1 - x^2}y = \frac{1}{1 - x^2}.$$

Integrating factor:

$$I = \exp\left(\int p(x) dx\right) = \exp\left(\int \frac{-x}{1 - x^2} dx\right) = \exp\left(\frac{1}{2} \ln(1 - x^2)\right) = (1 - x^2)^{1/2}.$$

Multiplying the ODE by the integrating factor transforms it into

$$\frac{d}{dx} (y(1 - x^2)^{1/2}) = \frac{1}{(1 - x^2)^{1/2}}.$$

[Check this by differentiating out the LHS if you don't believe it.] Hence,

$$y(1 - x^2)^{1/2} = \int \frac{1}{(1 - x^2)^{1/2}} dx = \arcsin x + C,$$

so
$$y = \frac{\arcsin x + C}{(1 - x^2)^{1/2}}.$$

This is the general solution.

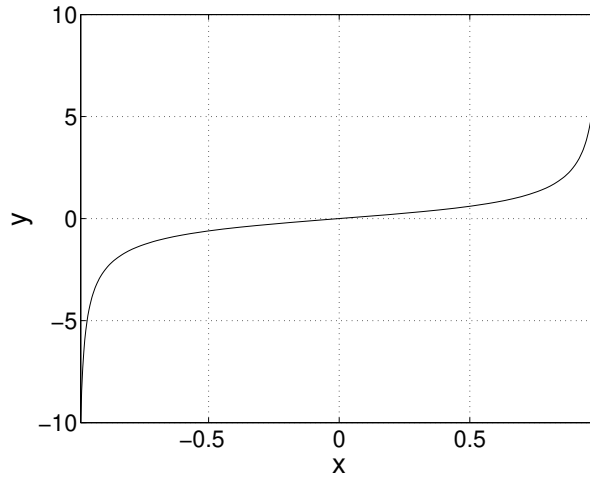
Initial conditions: We are given that $y = 0$ at $x = 0$. Substituting these values into the general solution, we get:

$$0 = \frac{0 + C}{1} \implies C = 0.$$

So the required solution is

$$y = \frac{\arcsin x}{(1 - x^2)^{1/2}}.$$

This is valid for $-1 < x < 1$:



Note that the solution is singular where the term multiplying dy/dx in (2) vanishes.

(b)

$$\frac{dy}{dx} - \frac{y}{x} = x \cos x$$

is a linear first-order ODE – already in its standard form with $p(x) = -\frac{1}{x}$.

Integrating factor, $I = \exp(\int p(x) dx)$,

$$I = \exp(-\ln x) = \frac{1}{x}.$$

Multiplying the ODE by the integrating factor transforms it into

$$\frac{d}{dx} \left(\frac{y}{x} \right) = \cos x.$$

So,

$$\begin{aligned} \frac{y}{x} &= \sin x + C \\ y &= x \sin x + C x. \end{aligned}$$

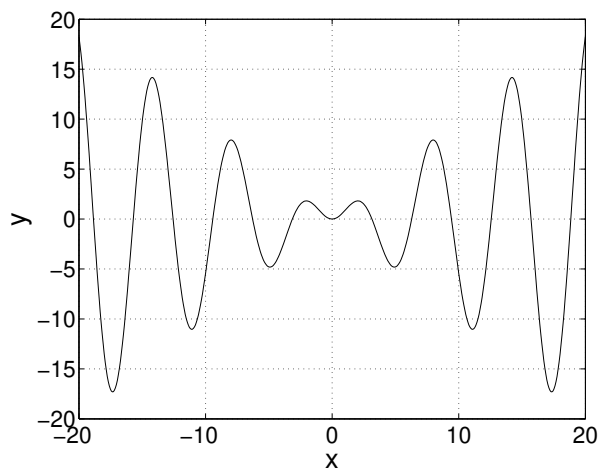
This is the general solution.

Initial conditions: We are given that $y(\pi) = 0$. Substituting these values into the general solution, we get:

$$0 = 0 + C \pi \implies C = 0.$$

So the required solution is

$$y = x \sin x.$$



The solution is defined for all values of x .