MATH10222: SOLUTIONS TO EXAMPLE SHEET[1] II

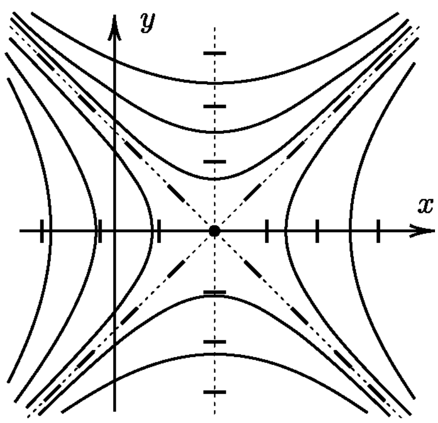
1. **Existence, uniqueness and graphical solutions**

   (a) To apply the existence and uniqueness theorem, rewrite the ODE in its standard from $y' = f(x, y)$. The existence and uniqueness theorem guarantees the existence of a unique solution in the vicinity of the point (X, Y) if $f(x, y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous functions of x and y in the vicinity of (X, Y).
   For our ODE,
   $$f(x, y) = \frac{x-1}{y}$$
   and
   $$\frac{\partial f(x, y)}{\partial y} = -\frac{x-1}{y^2},$$
   therefore the existence of a unique solution in the vicinity of (X, Y) is guaranteed for all $\{(X, Y) \,|\, Y \neq 0\}$.
   The ODE is nonlinear, therefore the existence and uniqueness theorem *only* ensures the existence in the vicinity of (X, Y), not for all values of x.

   (b) Isoclines (lines along which the solution of the ODE has the same slope) are given by $y' = (x-1)/y = c$, a constant. Thus the isocline on which the solution has slope c is given by $y_{iso} = (x-1)/c$. These are straight lines passing through $(x, y) = (1, 0)$ with slope $1/c$. Here are a few "obvious" ones:

   - $y' = 0$ on the vertical line $x = 1$.
   - $y' = \infty$ on the horizontal line $y = 0$, i.e. on the x-axis.
   - $y' = 1$ on $y = x - 1$
   - $y' = -1$ on $y = -(x - 1)$

   Here's a sketch of these isoclines and the corresponding integral curves:



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There's a critical point at $(x, y) = (1, 0)$ where the isoclines intersect.

All solution curves appear to approach the asymptotes $y = \pm(x-1)$ as $x \to \pm\infty$.

(c) The ODE is separable:

$$y\frac{dy}{dx} = x - 1,$$

$$\int y\ dy = \int (x-1)\ dx,$$

$$\frac{1}{2}y^2 = \frac{1}{2}(x-1)^2 + A \qquad \text{for any constant } A,$$

$$y = \pm\sqrt{(x-1)^2 + C} \qquad \text{for any constant } C\ (= 2A).$$

(d)
- As $x \to \pm\infty$, we have $(x-1)^2 \gg |C|$ for any (finite) value of the constant C so the lines $y = \pm(x-1)$ are indeed asymptotes for all solutions.
- For $C = 0$, we obtain two solutions $y = \pm(x-1)$ – the two asymptotes that emerge from the critical point.
- If $C > 0$, the solution curves pass through the line $x = 1$ at either $y = \sqrt{C}$ or $y = -\sqrt{C}$, corresponding the solutions above or below the critical point.
- If $C < 0$ the (real) solutions can't reach $x = 1$ – the solutions intersects the x-axis with infinite slope at $x = 1 \pm \sqrt{-C}$. These correspond to the solution to the right and left of the critical point.

(e) Existence and uniqueness was guaranteed, at least locally, if $Y \neq 0$. The sketch shows what goes wrong if we apply initial conditions on the x-axis: For each initial condition of the form $y(x = X) = 0$, there are two possible solutions – one with $y \geq 0$, the other one with $y \leq 0$.

Regarding the existence of solutions: Recall that for nonlinear ODEs the existence and uniqueness theorem only provides local results: Existence of the solution close to the initial conditions does not ensure its existence for all values of x. In our example, consider the family of solutions that cross the y-axis, i.e. those with initial conditions of the form $y(x = 0) = Y$. While the solutions for $|Y| > 1$ exist for all values of x, those for $|Y| < 1$ only exist over a limited range of x-values, up to the point where they intersect the x-axis.

2. **Separable ODEs**

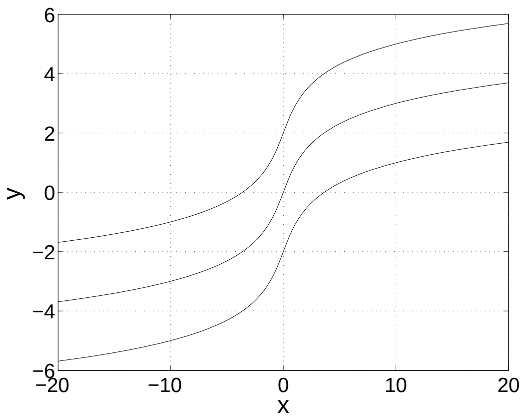
(a)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

Separate and integrate

$$\int dy = y = \int \frac{1}{\sqrt{1+x^2}}\, dx + C = \operatorname{arcsinh} x + C.$$

This is the general solution. Here's a plot of the solution for various values of the constant C.



The solution curves all have the same shape. Variations in C shift them along the y-axis.

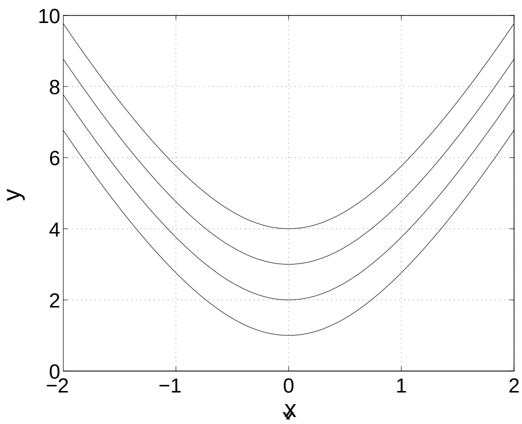
(b)

$$\frac{dy}{dx} = \frac{4\,x}{(1+x^2)^{1/3}}$$

Separate and integrate, using the substitution $z = 1 + x^2$. This yields

$$y = 3\,(1+x^2)^{2/3} + C.$$

Here's a sketch of the solutions:



Again, the constant C simply shifts the position of the solution curves.

(c)

$$\frac{dy}{dx} = \frac{-2\,y}{x-2}$$

Observations: (i) $y \equiv 0$ is a solution. **(ii)** If $y_1(x)$ is a solution of the ODE then $y_2(x) = -y_1(x)$ is a solution, too.
Separate

$$\frac{1}{y}\frac{dy}{dx} = -\frac{2}{x-2} \qquad \text{for } y \neq 0$$

(Note that we've dealt with the case $y = 0$ already: It's also a solution!) and integrate

$$\int \frac{1}{y}\,dy = -\int \frac{2}{x-2}\,dx.$$

$$\ln|y| = -2\,\ln|x-2| + C$$

for any constant C. Rewrite

$$\ln|y| = \ln|x-2|^{-2} + \ln|K|,$$

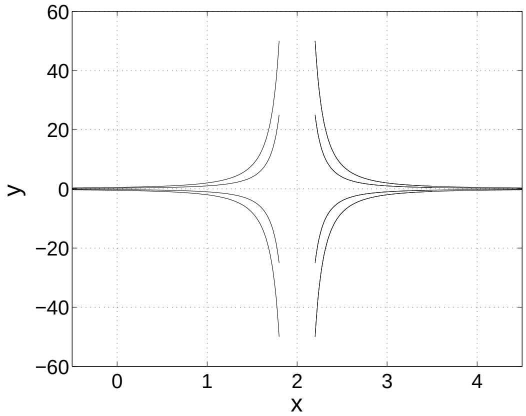
for another constant, K, and combine the logarithms:

$$\ln\left|\frac{y\,(x-2)^2}{K}\right| = 0 \qquad \text{only for } K \neq 0$$

so

$$y = \frac{K}{(x-2)^2} \qquad \text{for } K \in \mathbb{R} \text{ since } y \equiv 0 \text{ is a solution too!}$$

The arbitrary constant K **multiplies** the function. If we change K the **shape** of the solution changes.



Note that the solution $y \equiv 0$ is an asymptote for all solutions as $x \to \pm\infty$.

(d)

$$\sqrt{1+x^2}\,\frac{dy}{dx} = y$$

Observations: **(i)** $y \equiv 0$ is a solution. **(ii)** If $y_1(x)$ is a solution of the ODE then $y_2(x) = -y_1(x)$ is a solution, too.
Separate

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \qquad \text{for } y \neq 0,$$

and integrate

$$\int \frac{1}{y}\,dy = \int \frac{dx}{\sqrt{1+x^2}}$$
$$\ln|y| = \operatorname{arcsinh} x + C$$

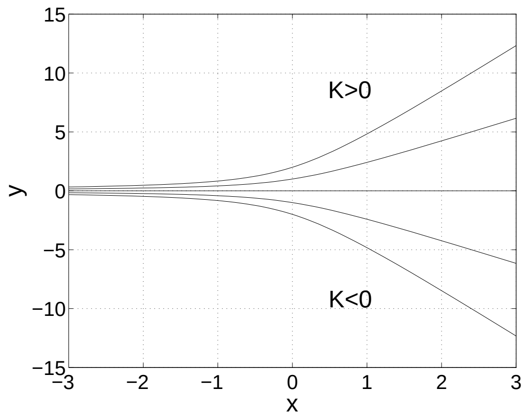
Rewrite, using the hint,

$$\ln|y| = \ln(x+\sqrt{1+x^2}) + \ln|K| = \ln|K\,(x+\sqrt{1+x^2})|$$

so

$$y = K\,(x+\sqrt{1+x^2}) \qquad \text{for } K \in \mathbb{R}$$

since $y \equiv 0$ is also a solution.
Here is a sketch of the solution



As in the previous example, the constant of integration changes the shape of the solution. The solution $y \equiv 0$ is an asymptote for $x \to -\infty$.

3. **Initial value problems**

   (a) We have calculated the general solution of the ODE in question 2a:

   $$y(x) = \operatorname{arcsinh} x + C$$

   Applying the initial condition $y(0) = 5$ yields $5 = \operatorname{arcsinh} 0 + C = C$ so the solution of the initial value problem is

   $$y(x) = \operatorname{arcsinh} x + 5.$$

   (b) We have calculated the general solution of the ODE in question 2d:

   $$y = K\,(x + \sqrt{1 + x^2})$$

   Applying the initial condition $y(0) = -3$ yields $-3 = K\,(0 + \sqrt{1+0}) = K$ so the solution of the initial value problem is

   $$y = -3\,(x + \sqrt{1 + x^2}).$$

4. **First-order ODEs of homogeneous type**

   (a)

   $$x\,y\,\frac{dy}{dx} + x^2 + y^2 = 0. \tag{1}$$

   Assuming that $x \neq 0, y \neq 0$, we rewrite this as

   $$\frac{dy}{dx} = -\frac{x}{y} - \frac{y}{x},$$

   which shows that the equation is a first-order ODE of homogeneous type. Put $y(x) = z(x)\,x$, thus $\frac{dy}{dx} = z + x\,\frac{dz}{dx}$. The ODE becomes

   $$z + x\,\frac{dz}{dx} = -\frac{1}{z} - z = -\frac{1 + z^2}{z}$$

i.e.

$$x\,\frac{dz}{dx} = -\frac{1+2\,z^2}{z}.$$

Separate

$$\frac{z}{1+2\,z^2}\,\frac{dz}{dx} = -\frac{1}{x}$$

$$\int \frac{z}{1+2\,z^2}\,dz = -\int \frac{1}{x}\,dx$$

[Use the substitution $u = 1 + 2z^2$]

$$\frac{1}{4}\,\ln|1+2\,z^2| = -\ln|x| + C$$

$$\ln|1+2\,z^2|^{\frac{1}{4}} = -\ln|x| + \ln|K| = \ln|K/x|$$

$$1+2\,z^2 = \left(\frac{K}{x}\right)^4$$

$$2\,\frac{y^2}{x^2} = \left(\frac{K}{x}\right)^4 - 1$$

$$y = \pm x\sqrt{\frac{1}{2}\left(\left(\frac{K}{x}\right)^4 - 1\right)}$$

This is the general solution for $x \neq 0$. Note that for $x = 0$ the coefficient multiplying dy/dx in (1) vanishes – this is always a sign of trouble!

(b)

$$x^2\,\frac{dy}{dx} + y^2 - xy = 0$$

Observation: $y \equiv 0$ is a solution.

Rewriting the ODE as

$$\frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2}$$

shows that the equation is a first-order ODE of homogeneous type.

Put $y(x) = z(x)\,x$, thus $\frac{dy}{dx} = z + x\,\frac{dz}{dx}$. The ODE becomes

$$z + x\,\frac{dz}{dx} = z - z^2$$

i.e.

$$x\,\frac{dz}{dx} = -z^2.$$

Separate,

$$-\frac{1}{z^2}\,\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{1}{z} = \ln|x| + C$$

$$\frac{1}{y} = \frac{\ln|x| + C}{x} \qquad (x \neq 0,\ y \neq 0)$$

$$y = \frac{x}{\ln|x| + C}$$

This is the general solution for $x \neq 0, y \neq 0$. We know that $y \equiv 0$ is another solution. At $x = 0$ the RHS of the ODE is singular and the solution is not defined.

5. **First-order linear ODEs**

(a)

$$(1 - x^2)\frac{dy}{dx} - x\,y = 1 \tag{2}$$

is a linear first-order ODE.
Rearrange into the standard form $dy/dx + p(x)\,y(x) = q(x)$:

$$\frac{dy}{dx} - \frac{x}{1 - x^2}\,y = \frac{1}{1 - x^2}.$$

Integrating factor:

$$I = \exp\left(\int p(x)\,dx\right) = \exp\left(\int \frac{-x}{1 - x^2}\,dx\right) = \exp\left(\frac{1}{2}\ln(1 - x^2)\right) = (1-x^2)^{1/2}.$$

Multiplying the ODE by the integrating factor transforms it into

$$\frac{d}{dx}\left(y\,(1 - x^2)^{1/2}\right) = \frac{1}{(1 - x^2)^{1/2}}.$$

[Check this by differentiating out the LHS if you don't believe it.] Hence,

$$y\,(1 - x^2)^{1/2} = \int \frac{1}{(1 - x^2)^{1/2}}\,dx = \arcsin x + C,$$

so

$$y = \frac{\arcsin x + C}{(1 - x^2)^{1/2}}.$$

This is the general solution.

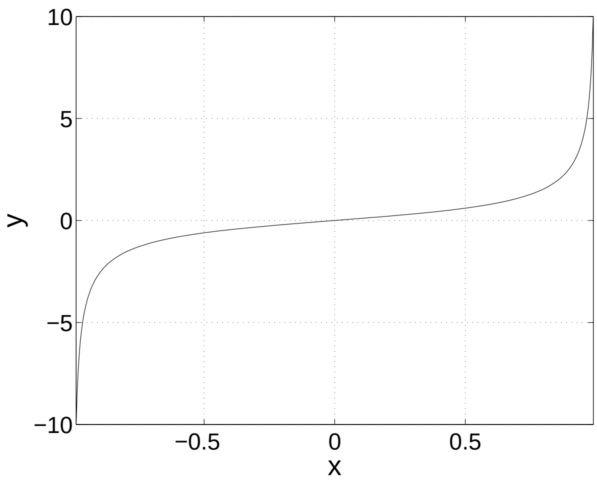
<u>Initial conditions</u>: We are given that $y = 0$ at $x = 0$. Substituting these values into the general solution, we get:

$$0 = \frac{0 + C}{1} \quad \Longrightarrow \quad C = 0.$$

So the required solution is

$$y = \frac{\arcsin x}{(1 - x^2)^{1/2}}.$$

This is valid for $-1 < x < 1$:



Note that the solution is singular where the term multiplying dy/dx in (2) vanishes.

(b)

$$\frac{dy}{dx} - \frac{y}{x} = x\,\cos x$$

is a linear first-order ODE – already in its standard form with $p(x) = -\frac{1}{x}$.
Integrating factor, $I = \exp(\int p(x)\,dx)$,

$$I = \exp(-\ln x) = \frac{1}{x}.$$

Multiplying the ODE by the integrating factor transforms it into

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \cos x.$$

So,

$$\frac{y}{x} = \sin x + C$$

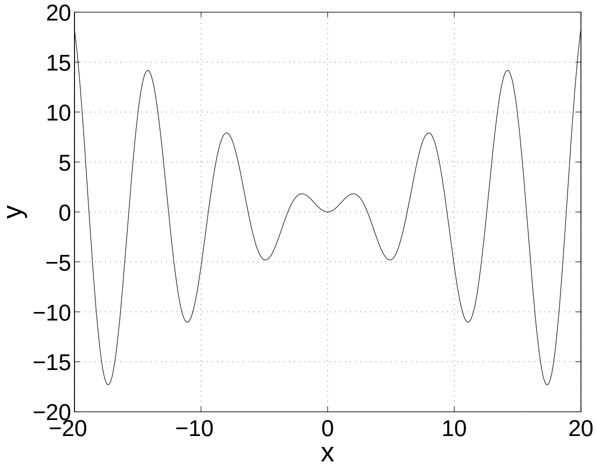
$$y = x\,\sin x + C\,x.$$

This is the general solution.
Initial conditions: We are given that $y(\pi) = 0$. Substituting these values into the general solution, we get:

$$0 = 0 + C\,\pi \quad \Longrightarrow \quad C = 0.$$

So the required solution is

$$y = x\,\sin x.$$



The solution is defined for all values of x.