MATH10222: SOLUTIONS TO EXAMPLE SHEET

1. Integration as the inversion of differentiation

(a) The derivative of \( y_1(x) = \sin(x) \)

is (obviously!)

\[ \frac{dy_1}{dx} = \cos(x) \]

and this answer is unique. Geometrically, this is because the slope of the (smooth) function \( \sin(x) \) is unique.

(b) Given the derivative of \( y_2(x) \) as

\[ \frac{dy_2}{dx} = \cos(x), \]

the function \( y_2(x) \) itself may be obtained by straightforward integration:

\[ \int \frac{dy_2}{dx} \, dx = \int \cos(x) \, dx \]

\[ y_2(x) = \sin(x) + C \]

where \( C \) is the (arbitrary!) constant of integration. The answer to the question is therefore not unique. There are infinitely many functions whose slope is given by \( \cos(x) \). The functions differ by constants.

(c) Given the second derivative of \( y_3(x) \) as

\[ \frac{d^2y_3}{dx^2} = -\sin(x) \]

we may again obtain the function \( y_3(x) \) by integration:

\[ \int \frac{d^2y_3}{dx^2} \, dx = \int -\sin(x) \, dx, \]

\[ \frac{dy_3}{dx} = \cos(x) + C, \]

where \( C \) is an arbitrary constant of integration. Applying this procedure again yields

\[ \int \frac{dy_3}{dx} \, dx = \int (\cos(x) + C) \, dx, \]

\[ y_3(x) = \sin(x) + Cx + D, \]

where \( D \) is another arbitrary constant of integration.

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The answer to the question is therefore not unique. This is because we were given the second derivative of \( y_3(x) \) and therefore had to integrate twice to obtain the function itself. In the process two constants of integration were introduced. Compared to the (arguably) simplest solution \( y_3(x) = \hat{y}_3(x) = \sin(x) \), the other solutions may be obtained by

- adding arbitrary constants, and/or
- adding arbitrary multiples of the function \( y(x) = x \).

2. Ordinary Differential Equations (ODEs) as relations between a function and its derivatives

(a) You should know that the derivative of \( y(x) = \exp(x) \) is \( y'(x) = \exp(x) = y(x) \) again, so \( y_4(x) = \exp(x) \) is a solution of

\[
\frac{dy_4}{dx} = y_4(x).
\]

However, this is clearly not the only solution as any multiple of \( \exp(x) \) also satisfies the above equation. The answer is therefore not unique. We will soon show that

\[
y_4(x) = A \exp(x)
\]

where \( A \) is an arbitrary constant is, in fact, the most general solution.

(b) You should know that the second derivative of \( y(x) = \cos(x) \) is \( y''(x) = -\cos(x) = -y(x) \) so \( y_5(x) = \cos(x) \) is a solution of

\[
\frac{d^2y_5}{dx^2} = -y_5(x).
\]

As in the previous question, it should be obvious that this solution is not unique, as \( y_5(x) = A \cos(x) \) also solves the above equation for any value of the constant \( A \). Furthermore, exactly the same arguments also apply to \( y(x) = \sin(x) \), so an even more general solution is given by

\[
y_5(x) = A \cos(x) + B \sin(x),
\]

where \( A \) and \( B \) are arbitrary constants. We will soon show that this is, in fact, the most general solution.