MATH10222: SOLUTIONS TO EXAMPLE SHEET[1] 0

1. **Integration as the inversion of differentiation**

   (a) The derivative of
   $$y_1(x) = \sin(x)$$
   is (obviously!)
   $$\frac{dy_1}{dx} = \cos(x)$$
   and this answer is unique. Geometrically, this is because the slope of the (smooth) function $\sin(x)$ is unique.

   (b) Given the derivative of $y_2(x)$ as
   $$\frac{dy_2}{dx} = \cos(x),$$
   the function $y_2(x)$ itself may be obtained by straightforward integration:
   $$\int \frac{dy_2}{dx}\, dx = \int \cos(x) dx$$
   $$y_2(x) = \sin(x) + C$$
   where C is the (arbitrary!) constant of integration. The answer to the question is therefore not unique. There are infinitely many functions whose slope is given by $\cos(x)$. The functions differ by constants.

   (c) Given the second derivative of $y_3(x)$ as
   $$\frac{d^2y_3}{dx^2} = -\sin(x)$$
   we may again obtain the function $y_3(x)$ by integration:
   $$\int \frac{d^2y_3}{dx^2}\, dx = \int -\sin(x) dx,$$
   $$\frac{dy_3}{dx} = \cos(x) + C,$$
   where C is an arbitrary constant of integration. Applying this procedure again yields
   $$\int \frac{dy_3}{dx}\, dx = \int (\cos(x) + C)\, dx,$$
   $$y_3(x) = \sin(x) + Cx + D,$$
   where D is another arbitrary constant of integration.

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The answer to the question is therefore not unique. This is because we were given the second derivative of $y_3(x)$ and therefore had to integrate twice to obtain the function itself. In the process two constants of integration were introduced. Compared to the (arguably) simplest solution $y_3(x) = \widehat{y_3}(x) = \sin(x)$, the other solutions may be obtained by

- adding arbitrary constants, and/or
- adding arbitrary multiples of the function $y(x) = x$.

2. **Ordinary Differential Equations (ODEs) as relations between a function and its derivatives**

   (a) You should know that the derivative of $y(x) = \exp(x)$ is $y'(x) = \exp(x) = y(x)$ again, so $y_4(x) = \exp(x)$ is *a* solution of

   $$\frac{dy_4}{dx} = y_4(x).$$

   However, this is clearly not the only solution as any multiple of $\exp(x)$ also satisfies the above equation. The answer is therefore not unique. We will soon show that

   $$y_4(x) = A \exp(x)$$

   where A is an arbitrary constant is, in fact, *the* most general solution.

   (b) You should know that the second derivative of $y(x) = \cos(x)$ is $y''(x) = -\cos(x) = -y(x)$ so $y_5(x) = \cos(x)$ is *a* solution of

   $$\frac{d^2y_5}{dx^2} = -y_5(x).$$

   As in the previous question, it should be obvious that this solution is not unique, as $y_5(x) = A\cos(x)$ also solves the above equation for any value of the constant A. Furthermore, exactly the same arguments also apply to $y(x) = \sin(x)$, so *an* even more general solution is given by

   $$y_5(x) = A\cos(x) + B\sin(x),$$

   where A and B are arbitrary constants. We will soon show that this is, in fact, *the* most general solution.