Some theory for *linear* 2nd order ODEs

Existence and Uniqueness

Consider the *linear* second-order ODE

\[ y'' + p(x) y' + q(x) y = r(x), \]  

subject to the initial conditions

\[ y(X) = Y, \quad y'(X) = Z, \]  

where the constants \( X, Y \) and \( Z \), and the functions \( p(x) \), \( q(x) \) and \( r(x) \) are given.

**Theorem**

If the functions \( p(x) \), \( q(x) \) and \( r(x) \) are continuous functions of \( x \) in an interval \( I \), and if \( X \in I \) then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in the entire interval \( I \).

**Notes:**

- This is the promised extension of the statement for first-order problems. The extension to even higher-order linear ODEs should be obvious...

- If the functions \( p(x) \), \( q(x) \) and \( r(x) \) are “well-behaved” (no jumps, singularities, etc.), the theorem guarantees the existence of a unique solution for \( x \in \mathbb{R} \).

- However, the statement still only applies to initial value problems!
The homogeneous ODE & superposition of its solutions

If we set \( r(x) = 0 \) in the \textit{inhomogeneous} ODE
\[
y'' + p(x) y' + q(x) y = r(x), \quad (I)
\]
we obtain the corresponding \textit{homogeneous} ODE
\[
y'' + p(x) y' + q(x) y = 0. \quad (H)
\]

A trivial (?) but useful observation

If \( y_1(x) \) and \( y_2(x) \) are two solutions of \((H)\) then the linear combination
\[
y_3(x) = A y_1(x) + B y_2(x)
\]
is also a solution, for any values of the constants \( A \) and \( B \).

Linear independence

To see why this is a useful observation, we need to define the concept of linear independence: Two nonzero functions \( y_1(x) \) and \( y_2(x) \) are linearly independent if
\[
A y_1(x) + B y_2(x) = 0 \quad \forall x \quad \iff \quad A \equiv B \equiv 0
\]
(...just as in linear algebra...).

Examples:

- \( y_1(x) = x \) and \( y_2(x) = 3 x^2 \) are linearly independent.
- \( y_1(x) = x \) and \( y_2(x) = 3 x \) are linearly dependent – they’re just multiples of each other.
Fundamental solutions of the homogeneous ODE

Theorem

Any solution of the homogeneous ODE

\[ y'' + p(x) y' + q(x) y = 0. \]  \hspace{1cm} (H)

can be written as a linear combination of any two non-zero, linearly independent solutions, \( y_1(x) \) and \( y_2(x) \), say:

\[ y(x) = A y_1(x) + B y_2(x). \]

The two non-zero, linearly independent solutions \( \{ y_1(x), y_2(x) \} \) are called “fundamental solutions” of the homogeneous ODE (H).

Notes:

- The set of fundamental solutions is not unique!
The general solution of the inhomogeneous ODE

Theorem

The general solution of the inhomogeneous ODE

\[ y'' + p(x) y' + q(x) y = r(x) \]  \hspace{1cm} (I)

can be written as

\[ y(x) = y_p(x) + A y_1(x) + B y_2(x), \]

where:

- \( A \) and \( B \) are arbitrary constants.
- \( y_p(x) \) is any particular solution of the inhomogeneous ODE.
- \( y_1(x) \) and \( y_2(x) \) are fundamental solutions of the corresponding homogeneous ODE.

Notes:

- Note the similarities between the structure of the solution of the linear ODE and the structure of the solution of the linear (algebraic) equation \( Ax = b \). This is not accidental! There are deep connections between the two fields – matrices and the homogeneous part of a linear ODE are both “linear operators”.
- The values of the constants \( A \) and \( B \) are determined by the boundary or initial conditions.