Some theory for *linear* 2nd order ODEs

Existence and Uniqueness

Consider the *linear* second-order ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x), \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the functions $p(x)$, $q(x)$ and $r(x)$ are given.

Theorem

> If the functions $p(x)$, $q(x)$ and $r(x)$ are continuous functions of x in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in the entire interval I.

Notes:

- This is the promised extension of the statement for first-order problems. The extension to even higher-order linear ODEs should be obvious...
- If the functions $p(x)$, $q(x)$ and $r(x)$ are "well-behaved" (no jumps, singularities, etc.), the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

The homogeneous ODE & superposition of its solutions

If we set $r(x) = 0$ in the *inhomogeneous* ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x), \qquad \text{(I)}$$

we obtain the corresponding *homogeneous* ODE

$$y'' + p(x)\,y' + q(x)\,y = 0. \qquad \text{(H)}$$

A trivial (?) but useful observation

If $y_1(x)$ and $y_2(x)$ are two solutions of (H) then the linear combination

$$y_3(x) = A\,y_1(x) + B\,y_2(x)$$

is also a solution, for any values of the constants A and B.

Linear independence

To see why this is a useful observation, we need to define the concept of linear independence: Two nonzero functions $y_1(x)$ and $y_2(x)$ are linearly independent if

$$A\,y_1(x) + B\,y_2(x) = 0 \quad \forall x \qquad \Longleftrightarrow \qquad A \equiv B \equiv 0$$

(...just as in linear algebra...).

Examples:

- $y_1(x) = x$ and $y_2(x) = 3\,x^2$ are linearly independent.
- $y_1(x) = x$ and $y_2(x) = 3\,x$ are linearly dependent – they're just multiples of each other.

Fundamental solutions of the homogeneous ODE

Theorem

Any solution of the homogeneous ODE

$$y'' + p(x)\, y' + q(x)\, y = 0. \qquad \text{(H)}$$

can be written as a linear combination of *any* two non-zero, linearly independent solutions, $y_1(x)$ and $y_2(x)$, say:

$$y(x) = A\, y_1(x) + B\, y_2(x).$$

The two non-zero, linearly independent solutions $\{y_1(x),\, y_2(x)\}$ are called "fundamental solutions" of the homogeneous ODE (H).

Notes:

- The set of fundamental solutions is not unique!

The general solution of the inhomogeneous ODE

Theorem

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + A\,y_1(x) + B\,y_2(x),$$

where:

- A and B are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Notes:

- Note the similarities between the structure of the solution of the linear ODE and the structure of the solution of the linear (algebraic) equation $\mathbf{A}\mathbf{x} = \mathbf{b}$. This is not accidental! There are deep connections between the two fields – matrices and the homogeneous part of a linear ODE are both "linear operators".
- The values of the constants A and B are determined by the boundary or initial conditions.