Basic ideas of perturbation methods: "Exploiting small parameters" and "Scaling"

Observation 1:

- ODEs (and hence their solutions!) typically contain some parameters, e.g.

$$m\ddot{x} + k\dot{x} + cx = F\ \cos(\Omega t)$$

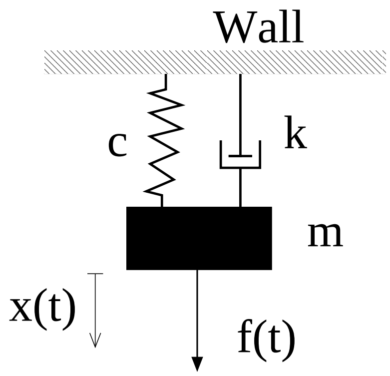
so

$$x = x(t) = x(t; m, k, c, \Omega).$$

- Often some of the problem's parameters are "small". How can we exploit this?
- Example:
  - Assume that we (only) know the solution of the above ODE for $k = 0$ (no damping).
  - What is the solution for "small" k?

Observation 2:

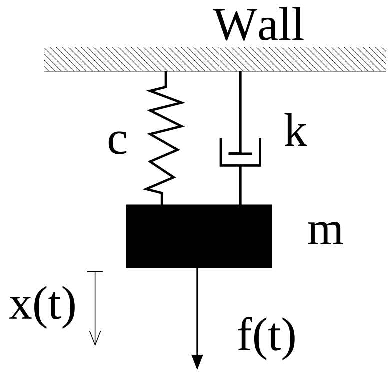
- ODEs that model physical phenomena typically express balances (of forces, energies, currents, ...).
- Here's an example of a balance of forces:



$$\underbrace{m\ddot{x}}_{\text{inertial forces}} + \underbrace{k\dot{x}}_{\text{damping forces}} + \underbrace{cx}_{\text{spring forces}} = \underbrace{F\ \cos(\Omega t)}_{\text{applied external force}}$$

- In general, all terms in the ODE will make a significant contribution to the overall "balance".
- However, there *may* be regimes in which the balance of terms is dominated by a balance between just a few (ideally two) terms, while the other terms only provide "negligible" contributions.
- The simplified equations (obtained by neglecting the small terms) are often much easier to solve than the full equations.
- We may [should!] then be interested in finding the effect that the "small" perturbations have on the solution.
- A seemingly trivial observation: You will need *at least* two terms to balance!

Example:



$$m\ddot{x} + k\dot{x} + cx = F\ \cos(\Omega t)$$

- We established earlier that

$$x(t) = x_P(t) + x_H(t)$$

where $x_H(t) \to 0$ very rapidly.

- Following the decay of the initial transients [described by $x_H(t)$] we have

$$x(t) \approx x_P(t) = A\ \cos(\Omega t) + B\ \sin(\Omega t)$$

- Hence if Ω is "small", the mass will move very slowly, implying that $m\ddot{x}$ and $k\dot{x}$ will be much smaller than cx.

- In this "quasi-steady" regime, we expect the motion of the mass to be described (approximately!) by

$$c\,x(t) \approx F\ \cos(\Omega t).$$

“Proof”

- Check that
$$x(t) \approx \frac{F}{c}\,\cos(\Omega t)$$
is an approximate solution of
$$m\ddot{x} + k\dot{x} + cx = F\,\cos(\Omega t)$$
if Ω is small.

- The exact solution is
$$x(t) \approx x_P(t) = A\,\cos(\Omega t) + B\,\sin(\Omega t)$$
where
$$A = F\,\frac{c - m\Omega^2}{(k\Omega)^2 + (c - m\Omega^2)^2} \quad \to \frac{F}{c} \quad \text{as } \Omega \to 0,$$
and
$$B = F\,\frac{k\Omega}{(k\Omega)^2 + (c - m\Omega^2)^2} \quad \to 0 \quad \text{as } \Omega \to 0.$$

“Q.E.D.”

Observation 3a:

- Coefficients occurring in ODEs that model physical phenomena have dimensions!
- The dimensions of all terms must be (are!) consistent.

$$m\ddot{x} + k\dot{x} + cx = F\ \cos(\Omega t)$$

$$\underbrace{m}\underbrace{\ddot{x}} + \underbrace{k}\underbrace{\dot{x}} + \underbrace{c}\underbrace{x} + = \underbrace{F}\cos(\underbrace{\Omega}\underbrace{t})$$

$$\underbrace{m}_{\text{kg}}\ \underbrace{\ddot{x}}_{\text{m/sec}^2} + \underbrace{k}_{?}\ \underbrace{\dot{x}}_{\text{m/sec}} + \underbrace{c}_{\text{N/m}}\ \underbrace{x}_{\text{m}} + = \underbrace{F}_{\text{N}}\cos(\underbrace{\Omega}_{1/\text{sec}}\ \underbrace{t}_{\text{sec}})$$

- What's the dimension of k? For dimensional consistency:

$$[k] = \text{N}/(\text{m}/\text{sec})$$

or (since $N = \text{kg}\,\text{m}/\text{sec}^2$; see $m\ddot{x}$)

$$[k] = \text{k}/\text{sec}$$

- The arguments of all functions (e.g. $\cos\Omega t$) are dimensionless!

Observation 3b:

- The solution tends to depend on ratios of dimensional coefficients.
- The ratios provide an indication of:
  1. The relative size of the physical effects* represented by the corresponding terms.

$$m\ddot{x} + k\dot{x} + cx = f\ \cos(\Omega t)$$

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = F\ \cos(\Omega t)$$

     where

$$\delta = \frac{k}{2m} = \frac{\text{“Damping forces”}}{\text{“Inertia”}}$$

     and

$$\omega^2 = \frac{c}{m} = \frac{\text{“Spring forces”}}{\text{“Inertia”}}.$$

  2. Time/length-scales over which the relevant phenomena occur. E.g.

$$x(t) = e^{-\delta t}\left(A\cos(t\sqrt{\omega^2 - \delta^2}) + B\sin(t\sqrt{\omega^2 - \delta^2})\right),$$

     showing that

     $\Longrightarrow$ $1/\delta$ is a representative timescale over which the oscillations decay.

     $\Longrightarrow$ $1/\omega$ is a representative timescale for the undamped oscillation.

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*: **Disclaimer:** Statement 1 is a bit too simple-minded – we might (!) have time to come back it...

Observations about Observations 1, 2 and 3

- The approach outlined above exploits *additional* knowledge about the problem.
- You will either have such knowledge *a priori* or you can make certain (hopefully plausible) assumptions about certain properties of the solution.
- In the latter case, you'll have to check the consistency of your assumptions when you're done. For instance:
  - Assume the the solution is such that certain terms in the ODE are small.
  - Neglect the small terms in the ODE and solve.
  - Check afterwards that the terms that were *assumed* to be small are *actually* small.
- The approach tends to produce approximate solutions of the ODE that are valid only in certain *"regions of parameter space"*, e.g. for small forcing frequencies Ω, small damping k, etc.
- This is often more useful than having an exact (but horrendously complicated) closed-form solution that is valid for all parameter values.