Basic ideas of perturbation methods:
“Exploiting small parameters”
and “Scaling”

Observation 1:

- ODEs (and hence their solutions!) typically contain some parameters, e.g.
  \[ m\ddot{x} + k\dot{x} + cx = F \cos(\Omega t) \]
  so
  \[ x = x(t) = x(t; m, k, c, \Omega). \]

- Often some of the problem’s parameters are “small”. How can we exploit this?

- Example:
  - Assume that we (only) know the solution of the above ODE for \( k = 0 \) (no damping).
  - What is the solution for “small” \( k \)?
Observation 2:

- ODEs that model physical phenomena typically express balances (of forces, energies, currents, ...).
- Here’s an example of a balance of forces:

\[ m \ddot{x} + k \dot{x} + cx = F \cos(\Omega t) \]

- Inertial forces + damping forces + spring forces = applied external force

- In general, all terms in the ODE will make a significant contribution to the overall “balance”.
- However, there may be regimes in which the balance of terms is dominated by a balance between just a few (ideally two) terms, while the other terms only provide “negligible” contributions.
- The simplified equations (obtained by neglecting the small terms) are often much easier to solve than the full equations.
- We may [should!] then be interested in finding the effect that the “small” perturbations have on the solution.
- A seemingly trivial observation: You will need at least two terms to balance!
Example:

![Illustration of a mass-spring-damper system with forces and displacements labeled.]

\[ m \ddot{x} + k \dot{x} + cx = F \cos(\Omega t) \]

- We established earlier that
  \[ x(t) = x_P(t) + x_H(t) \]
  where \( x_H(t) \rightarrow 0 \) very rapidly.

- Following the decay of the initial transients \([described by} x_H(t)]\ we have
  \[ x(t) \approx x_P(t) = A \cos(\Omega t) + B \sin(\Omega t) \]

- Hence if \( \Omega \) is “small”, the mass will move very slowly, implying that \( m \ddot{x} \) and \( k \dot{x} \) will be much smaller than \( cx \).

- In this “quasi-steady” regime, we expect the motion of the mass to be described (approximately!) by
  \[ c x(t) \approx F \cos(\Omega t). \]
“Proof”

• Check that

\[ x(t) \approx \frac{F}{c} \cos(\Omega t) \]

is an approximate solution of

\[ m\ddot{x} + k\dot{x} + cx = F \cos(\Omega t) \]

if \( \Omega \) is small.

• The exact solution is

\[ x(t) \approx x_P(t) = A \cos(\Omega t) + B \sin(\Omega t) \]

where

\[ A = F \frac{c - m\Omega^2}{(k\Omega)^2 + (c - m\Omega^2)^2} \quad \rightarrow \frac{F}{c} \quad \text{as} \quad \Omega \rightarrow 0, \]

and

\[ B = F \frac{k\Omega}{(k\Omega)^2 + (c - m\Omega^2)^2} \quad \rightarrow 0 \quad \text{as} \quad \Omega \rightarrow 0. \]

“Q.E.D.”
Observation 3a:

- Coefficients occurring in ODEs that model physical phenomena have dimensions!
- The dimensions of all terms must be (are!) consistent.

\[ m\ddot{x} + k\dot{x} + cx = F \cos(\Omega t) \]

- What’s the dimension of \( k \)? For dimensional consistency:
  \[ [k] = \text{N}/(\text{m/} \text{sec}) \]
  or (since \( N = \text{kg m/} \text{sec}^2 \); see \( m\ddot{x} \))
  \[ [k] = \text{k/sec} \]

- The arguments of all functions (e.g. \( \cos \Omega t \)) are dimensionless!
Observation 3b:

- The solution tends to depend on ratios of dimensional coefficients.
- The ratios provide an indication of:
  
  1. The relative size of the physical effects* represented by the corresponding terms.

\[
m\ddot{x} + k\dot{x} + cx = f \cos(\Omega t)
\]

\[
\ddot{x} + 2\delta\dot{x} + \omega^2 x = F \cos(\Omega t)
\]

where

\[
d = \frac{k}{2m} = \text{“Damping forces”}
\]

and

\[
\omega^2 = \frac{c}{m} = \text{“Spring forces”}.
\]

2. Time/length-scales over which the relevant phenomena occur. E.g.

\[
x(t) = e^{-\delta t} \left( A \cos(t\sqrt{\omega^2 - \delta^2}) + B \sin(t\sqrt{\omega^2 - \delta^2}) \right)
\]

showing that

\[1/\delta \text{ is a representative timescale over which the oscillations decay.}\]

\[1/\omega \text{ is a representative timescale for the undamped oscillation.}\]

*: Disclaimer: Statement 1 is a bit too simple-minded – we might (!) have time to come back it...
Observations about Observations 1, 2 and 3

• The approach outlined above exploits additional knowledge about the problem.

• You will either have such knowledge a priori or you can make certain (hopefully plausible) assumptions about certain properties of the solution.

• In the latter case, you’ll have to check the consistency of your assumptions when you’re done. For instance:

  – Assume the the solution is such that certain terms in the ODE are small.

  – Neglect the small terms in the ODE and solve.

  – Check afterwards that the terms that were assumed to be small are actually small.

• The approach tends to produce approximate solutions of the ODE that are valid only in certain “regions of parameter space”, e.g. for small forcing frequencies $\Omega$, small damping $k$, etc.

• This is often more useful than having an exact (but horrendously complicated) closed-form solution that is valid for all parameter values.