Where have we (you!) seen $x = x_P + x_H$ before?

Recall:

The *general* solution of the inhomogeneous ODE

$$y'' + p(x) y' + q(x) y = r(x) \quad (I)$$

can be written as

$$y(x) = y_p(x) + \alpha y_1(x) + \beta y_2(x),$$

where:

- $\alpha$ and $\beta$ are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Compare this to the solution of the system of linear (algebraic) equations:

$$A x = b,$$

where $A$ is an $n \times n$ matrix, and $b$ a given vector of size $n$.

The general solution $x$ (another vector of size $n$) is given by

$$x = x_P + x_H$$

where

- $x_P$ is a(ny) particular solution of $A x = b$
- $x_H$ is the *general* solution of the homogeneous system $A x = 0$. 
Example

\[
\begin{pmatrix}
1 & -1 & 0 \\
2 & -2 & 0 \\
3 & -3 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
\]

Note that the matrix is singular, so \( Ax = 0 \) has non-trivial solutions!

- Transform into “triangular” form

\[
\begin{pmatrix}
1 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

showing that the RHS is consistent. We’re left with one equation for three unknowns.

- Set \( x_2 = \alpha \) and \( x_3 = \beta \), where \( \alpha \) and \( \beta \) are arbitrary constants.

- The general solution is: \( x_1 = 1 + \alpha \) and, of course, \( x_2 = \alpha \) and \( x_3 = \beta \).

- Rewrite in vector form:

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
+ \alpha
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}
+ \beta
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

- Note that

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
1 \\
3.1415
\end{pmatrix}
+ \alpha'
\begin{pmatrix}
-42.2 \\
-42.2 \\
1145.2
\end{pmatrix}
+ \beta'
\begin{pmatrix}
523.2 \\
523.2 \\
13.423
\end{pmatrix}
\]

is another (not so pretty) representation of the general solution.
The key features of both solutions are:

- $x_P$ and $x'_P$ solve the inhomogeneous equation.
- $x_H$ and $x'_H$ “span the null space” of $A$, i.e. they
  1. satisfy $Ax = 0$,
  2. are nonzero,
  3. are linearly independent.

“Off the record comment”:

In linear algebra it’s “easier” to overlook the additional solutions represented by $x_H$. In an ODE context, the fact that BCs [or ICs] have to be satisfied too, tends to provide an instant “reminder” that just having a particular solution of the ODE is not enough to solve the entire IVP/BVP.