(1)

Where have we (you!) seen $x = x_P + x_H$ before?

Recall:

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + \alpha\, y_1(x) + \beta\, y_2(x),$$

where:

- α and β are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Compare this to the solution of the system of linear (algebraic) equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{b}$ a given vector of size n.

The general solution $\mathbf{x}$ (another vector of size n) is given by

$$\mathbf{x} = \mathbf{x}_P + \mathbf{x}_H$$

where

- $\mathbf{x}_P$ is a(ny) particular solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$
- $\mathbf{x}_H$ is the *general* solution of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.

2

Example

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Note that the matrix is singular, so $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions!

- Transform into "triangular" form

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 - x_2 = 1$

$x_1 - \alpha = 1$

$x_1 = 1 + \alpha$

showing that the RHS is consistent. We're left with one equation for three unknowns.

- Set $x_2 = \alpha$ and $x_3 = \beta$, where α and β are arbitrary constants.
- The general solution is: $x_1 = 1 + \alpha$ and, of course, $x_2 = \alpha$ and $x_3 = \beta$.
- Rewrite in vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{x}_P} + \underbrace{\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{x}_H}$$

- Note that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 3.1415 \end{pmatrix}}_{\mathbf{x}'_P} + \underbrace{\alpha' \begin{pmatrix} -42.2 \\ -42.2 \\ 1145.2 \end{pmatrix} + \beta' \begin{pmatrix} 523.2 \\ 523.2 \\ 13.423 \end{pmatrix}}_{\mathbf{x}'_H}$$

is another (not so pretty) representation of the general solution.

The key features of both solutions are:

- $\mathbf{x}_P$ and $\mathbf{x}'_P$ solve the inhomogeneous equation.
- $\mathbf{x}_H$ and $\mathbf{x}'_H$ "span the null space" of $\mathbf{A}$, i.e. they
  1. satisfy $\mathbf{Ax} = \mathbf{0}$,
  2. are nonzero,
  3. are linearly independent.

"Off the record comment":

In linear algebra it's "easier" to overlook the additional solutions represented by $\mathbf{x}_H$. In an ODE context, the fact that BCs [or ICs] have to be satisfied too, tends to provide an instant "reminder" that just having *a* particular solution of the ODE is not enough to solve the entire IVP/BVP.

Summary:

General tasks for soln. of 2^{nd} order lin. ODEs:

1. Find 2 lin. indep. nonzero solns. of the homog. ODE
2. Find one particular soln of the full eqn.
3. Add & apply ICs.

Constant coefficient ODEs

$$\boxed{y'' + p y' + q y = r(x) \quad \text{where } p \,\&\, q \text{ are constants}}$$

(I) Solns of the homog ODE

$$y'' + p y' + q y = 0 \qquad (H)$$

[Aside: unique soln exists for $x \in \mathbb{R}$.]

Ansatz:

Idea: y, y' & y'' vary with x but terms in ODE have to add up to zero.
Try to find a soln that has the same x-dependence in every term.

<u>Ansatz:</u>

$$y = A e^{\lambda x}$$

where A & λ are arbitrary constants.

$$y' = A \lambda e^{\lambda x}$$

$$y'' = A \lambda^2 e^{\lambda x}$$

into ODE

$$\underbrace{A\lambda^2 e^{\lambda x}}_{y''} + p \underbrace{A\lambda e^{\lambda x}}_{y'} + q \underbrace{A e^{\lambda x}}_{y} \overset{!}{=} 0$$

$$A \underset{\substack{\downarrow \\ \text{never zero}}}{e^{\lambda x}} \underbrace{(\lambda^2 + p\lambda + q)}_{=0} \overset{!}{=} 0 \quad \forall x$$

$\Downarrow$

$A = 0 \Rightarrow$ trivial soln $y = 0$

$\Rightarrow$ choose λ such that

$$\lambda^2 + p\lambda + q = 0$$

"characteristic polynomial"

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$\Rightarrow$ Two solns of (H) are

$$y_1 = e^{\lambda_1 x} \qquad y_2 = e^{\lambda_2 x}$$

BUT: possibility of repeated & complex roots. $\Rightarrow$ 3 cases.

① $p^2 > 4q$: λ_1 & λ_2 are real & distinct

Then gen. soln. of (H) is

$$y_H = A e^{\lambda_1 x} + B e^{\lambda_2 x}.$$

(2) $p^2 = 4q$: Repeated roots

Our ansatz produces only one soln

$$y_1 = e^{\lambda x}$$

However: $y_2 = x e^{\lambda x}$

is a second, lin. indep. soln of (H).

Proof: Recall $\lambda = -\frac{p}{2}$

$$p = -2\lambda$$

Also: $q = \frac{1}{4} p^2 = \lambda^2$

Stick y_2 into ODE

$$y_2' = e^{\lambda x}(1 + \lambda x)$$

$$y_2'' = \lambda(2 + \lambda x) e^{\lambda x}$$

Into ODE:

$$e^{\lambda x}\Big(\underbrace{\lambda(2+\lambda x)}_{y''}+p\,\underbrace{(1+\lambda x)}_{y'}+q\,\underbrace{x}_{y}\Big)\overset{?}{=}0$$

$$p = -2\lambda \qquad q = \lambda^2$$

$$\cancel{2\lambda}+\cancel{\lambda^2 x}-\cancel{2\lambda}-\cancel{2\lambda^2 x}+\cancel{\lambda^2 x}\overset{?}{=}0$$

$$0=0 \quad \checkmark$$

<u>So:</u>

$$y(x) = Ae^{\lambda x} + Bxe^{\lambda x}$$

is the gen soln of (H).

③ $p^2 < 4q$: λ_{12} are complex conjugates

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$= -\frac{p}{2} \pm i\underbrace{\sqrt{q - \left(\frac{p}{2}\right)^2}}_{\text{real \& pos.}}$$

$$\lambda_{12} = \mu \pm i\omega$$

Gen. soln:

$$y = \hat{A}\, e^{(\mu + i\omega)x} + \hat{B}\, e^{(\mu - i\omega)x}$$

$$y = e^{\mu x}\left(\hat{A}\, e^{i\omega x} + \hat{B}\, e^{-i\omega x}\right)$$

want a real soln!

So when we apply the IC then $\hat{A}$ & $\hat{B}$ will turn out to be complex.

Recall:

$$e^{\pm i\omega x} = \cos(\omega x) \pm i \sin(\omega x)$$

$\Rightarrow$ we are really dealing with multiples of $\cos(\omega x)$ & $\sin(\omega x)$.

$\Rightarrow$ we can write the gen. real soln as:

$$y = e^{\mu x}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

where A & B are real constants.

SEE EX. SHEET III

Example:

$$y'' - 3y' + 2y = 0$$

$$y \sim e^{\lambda x}$$

into ODE:

$$e^{\lambda x}\Big(\underbrace{\lambda^2}_{y''} - 3\underbrace{\lambda}_{y'} + 2\cdot\underbrace{1}_{y}\Big) \stackrel{!}{=} 0 \quad \forall x$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{12} = +\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2}$$

$$= \frac{3}{2} \pm \sqrt{\frac{9-8}{4}}$$

$$= \frac{3}{2} \pm \frac{1}{2}$$

$$\lambda_1 = 2 \;;\quad \lambda_2 = 1$$

$$y_H = Ae^{2x} + Be^{x}.$$

Example:

$$y'' + 2y' + y = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0$$

$$\lambda_{12} = -1 \quad \text{repeated root!}$$

$$y = Ae^{-x} + Bxe^{-x}$$

Example:

$$y'' + 2y' + 5y = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_{12} = -1 \pm \sqrt{(-1)^2 - 5}$$

$$\lambda_{12} = \underbrace{-1}_{\mu = -1} \pm \underbrace{2}_{\omega = 2} i$$

$$y(x) = e^{-x}\left(\hat{A} e^{2ix} + \hat{B} e^{-2ix}\right)$$

$$y(x) = e^{-x}\left(A \cos(2x) + B \sin(2x)\right)$$

II Particular solns

$$y'' + p y' + q y = r(x)$$

Gen. soln:

$$y = \underbrace{y_p}_{?} + \underbrace{A y_1 + B y_2}_{\checkmark}$$

Strategy: Trial & error, guided by the form of $r(x)$.

$\Rightarrow$ "Method of undetermined coefficients" we'll illustrate the idea & the pitfalls for

$$y'' + p y' + q y = \underbrace{A e^{ax}}_{r(x)}$$

A, a are known constants

Given the form of $r(x)$, try

$$y = \underset{\substack{\uparrow \\ \text{undetermined coeffn.}}}{C} e^{\overset{\text{given!}}{a} x}$$

into ODE:

$$C \cancel{e^{ax}} \Big(\underbrace{a^2}_{y''} + p \underbrace{a}_{y'} + q \underbrace{1}_{y} \Big) \overset{!}{=} A \cancel{e^{ax}}$$

$$C = \underbrace{\frac{A}{a^2+pa+q}}_{\text{all known terms!}}$$

So a soln of the ODE is

$$y_p = \frac{A}{a^2+pa+q} e^{ax}$$

BUT what if $a^2+pa+q = 0$?

<u>Note:</u> this happens if a happens to be a root of the char. poly.