Where have we (you!) seen $x = x_P + x_H$ before?

Recall:

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + \alpha\, y_1(x) + \beta\, y_2(x),$$

where:

- α and β are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Compare this to the solution of the system of linear (algebraic) equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{b}$ a given vector of size n.

The general solution $\mathbf{x}$ (another vector of size n) is given by

$$\mathbf{x} = \mathbf{x}_P + \mathbf{x}_H$$

where

- $\mathbf{x}_P$ is a(ny) particular solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$
- $\mathbf{x}_H$ is the *general* solution of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.

Example

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Note that the matrix is singular, so $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions!

- Transform into "triangular" form

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

  showing that the RHS is consistent. We're left with one equation for three unknowns.

- Set $x_2 = \alpha$ and $x_3 = \beta$, where α and β are arbitrary constants.

- The general solution is: $x_1 = 1 + \alpha$ and, of course, $x_2 = \alpha$ and $x_3 = \beta$.

- Rewrite in vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{x}_P} \underbrace{+ \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{x}_H}$$

- Note that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 3.1415 \end{pmatrix}}_{\mathbf{x}'_P} \underbrace{+ \alpha' \begin{pmatrix} -42.2 \\ -42.2 \\ 1145.2 \end{pmatrix} + \beta' \begin{pmatrix} 523.2 \\ 523.2 \\ 13.423 \end{pmatrix}}_{\mathbf{x}'_H}$$

  is another (not so pretty) representation of the general solution.

The key features of both solutions are:

- $\mathbf{x}_P$ and $\mathbf{x}'_P$ solve the inhomogeneous equation.
- $\mathbf{x}_H$ and $\mathbf{x}'_H$ "span the null space" of $\mathbf{A}$, i.e. they
    1. satisfy $\mathbf{A}\mathbf{x} = \mathbf{0}$,
    2. are nonzero,
    3. are linearly independent.

"Off the record comment":

In linear algebra it's "easier" to overlook the additional solutions represented by $\mathbf{x}_H$. In an ODE context, the fact that BCs [or ICs] have to be satisfied too, tends to provide an instant "reminder" that just having *a* particular solution of the ODE is not enough to solve the entire IVP/BVP.

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

General task:

1. Find 2 lin. indep. solns. to the homog. ODE

$$y'' + p(x)\,y' + q(x)\,y = 0$$

2. Find a(ny) particular soln. of the full eqn.
3. Add & apply IC. or BC

Constant coefficient ODEs

$$y'' + p y' + q y = r(x)$$

where p & q are constants.

I) Soln. of the homog. ODE

$$y'' + p y' + q y = 0 \qquad (H)$$

[Soln. exists $\forall x$].

Idea: Try a soln for which each term in the ODE has the same x-dependence.

Ansatz: $y(x) = A e^{\lambda x}$

into ODE:

$y = A e^{\lambda x}$
$y' = A \lambda e^{\lambda x}$
$y'' = A \lambda^2 e^{\lambda x}$

$$A e^{\lambda x} \Big(\underbrace{\lambda^2}_{y''} + \underbrace{p\lambda}_{y'} + \underbrace{q}_{y} \Big) = 0$$

$\forall x$

One option: $A = 0$

$\Rightarrow$ trivial soln.

or $\lambda^2 + p\lambda + q = 0$

This is the "characteristic polynomial".

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

The two non-zero, lin. indep. solns. of (H) are

$$y_1(x) = e^{\lambda_1 x} \quad \& \quad y_2(x) = e^{\lambda_2 x}$$

<u>But:</u> possibility of complex roots & repeated roots.

$\Rightarrow$ 3 cases

① $p^2 > 4q$: λ_1 & λ_2 are real & distinct

General soln. of (H)

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

② $p^2 = 4q$: Repeated root

$$\lambda_1 = \lambda_2 = \lambda = -\frac{p}{2}$$

Our ansatz only produces one solution!

However $y_2(x) = x e^{\lambda x}$ is another nonzero lin. indep soln. of (H).

Proof: Note: $p = -2\lambda$

$$q = \frac{1}{4}p^2 = \lambda^2$$

$$y = x e^{\lambda x}$$
$$y' = e^{\lambda x}(1 + \lambda x)$$
$$y'' = e^{\lambda x}(\lambda(2 + \lambda x))$$

$$y'' + py' + qy = 0$$

$$e^{\lambda x}\Big(\underbrace{\lambda(2+\lambda x)}_{y''} + \underbrace{p}_{-2\lambda}\underbrace{(1+\lambda x)}_{y'} + \underbrace{q}_{\lambda^2} x\Big) \overset{?}{=} 0$$

$$\lambda(2+\lambda x) - 2\lambda(1+\lambda x) + \lambda^2 x \overset{?}{=} 0$$

$$\cancel{2\lambda} + \cancel{\lambda^2 x} - \cancel{2\lambda} - \cancel{2\lambda^2 x} + \cancel{\lambda^2 x} \overset{?}{=} 0 \quad \checkmark$$

So in this case

$$y(x) = Ae^{\lambda x} + Bxe^{\lambda x}$$

is the gen. soln. of (H).

③ $p^2 < 4q$: λ_{12} are complex conjugates

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$= \underbrace{-\frac{p}{2}} \pm i\sqrt{q - \left(\frac{p}{2}\right)^2}$$

$$\lambda_{12} = \mu \pm i\omega$$

$$y(x) = \hat{A}e^{(\mu+i\omega)x} + \hat{B}e^{(\mu-i\omega)x}$$

$$y(x) = e^{\mu x}\left(\hat{A}e^{i\omega x} + \hat{B}e^{-i\omega x}\right)$$

complex

If we want a real valued soln. then we have to allow $\hat{A}, \hat{B}$ to complex too

(SEE X-SHEET)

Since $e^{\pm i\omega x} = \cos(\omega x) \pm i\sin(\omega x)$
we are really dealing with
linear combinations of
$\sin(\omega x)$ & $\cos(\omega x)$

Therefore: Gen. soln. of (H)

$$y(x) = e^{\mu x}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

where μ & ω are the
real & imag parts of λ

<u>Example:</u> $y'' - 3y' + 2y = 0$

$$y \sim e^{\lambda x}$$

$$e^{\lambda x}(\underbrace{\lambda^2}_{y''} - \underbrace{3\lambda}_{y'} + \underbrace{2}_{y}) \stackrel{!}{=} 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{12} = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 2}$$

$$= \frac{3}{2} \pm \sqrt{\frac{9-8}{4}}$$

$$= \frac{3}{2} \pm \frac{1}{2}$$

$$\lambda_1 = 1 \qquad \lambda_2 = 2$$

Gen. soln:

$$\underline{y(x) = Ae^{x} + Be^{2x}}$$

Ex:

$$y'' + 2y' + y = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0$$

$$\lambda_{1,2} = -1$$

$$\underline{y(x) = Ae^{-x} + Bxe^{-x}}$$

Ex: $y'' + 2y' + 5y = 0$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_{12} = -1 \pm \sqrt{(-1)^2 - 5}$$

$$= \underbrace{-1}_{\mu} \pm \underbrace{2}_{\omega} i$$

$$y(x) = e^{-x}\left(\hat{A} e^{2ix} + \hat{B} e^{-2ix}\right)$$

$$y(x) = e^{-x}\left(A\cos(2x) + B\sin(2x)\right)$$

II Particular solns.

$$y'' + p y' + q y = r(x)$$

Gen. soln:

$$y(x) = y_p(x) + \underbrace{A y_1(x) + B y_2(x)}_{\checkmark}$$

$$\Downarrow$$

$$?$$

Strategy: Trial and error guided by the form of $r(x)$

$\Longrightarrow$ "Method of undetermined coefficients"

To illustrate the idea & the pitfalls consider

$$y'' + p y' + q y = A e^{ax}$$

where A & a are given constants!

Given the form of $r(x)$, III

try:

$$y = C e^{ax}$$

$\uparrow$ undetermined coefficient

Same a as in e^{ax}

into ODE:

$$\cancel{C e^{ax}}\Big(\underbrace{a^2}_{y''} + \underbrace{pa}_{y'} + \underbrace{q}_{y}\Big) \stackrel{!}{=} A\cancel{e^{ax}}$$

$$C = \frac{A}{a^2+pa+q}$$