(1)

Sep. ODE

$$g(y)\frac{dy}{dx} = h(x)$$

$$\int g(y(x))\frac{dy}{dx}\,dx = \int h(x)\,dx$$

$$\int g(y)\,dy = \int h(x)\,dx$$

Integr. by subst

$$\int f(y(x))\frac{dy}{dx}\,dx$$

$$\int f(y)\,dy$$; subst: $y = y(x)$

Example:

$$I = \int y \exp\left(-\frac{y^2}{2}\right) dy$$

subst:

$$x = -\frac{y^2}{2}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{-y}$$

$$I = \int y \exp(x) \frac{dy}{dx}\,dx$$

$$= \int \cancel{y} \exp(x) \frac{1}{-\cancel{y}}\,dx$$

$$= -\int \exp(x)\,dx = -\exp(x) + C$$

$$= -\exp\left(-\frac{y^2}{2}\right)$$

Linear 1st order ODEs

$$y' + p(x)\, y = q(x)$$

(integrating factor)

<u>Ex:</u> $$y' \underbrace{-x}_{p(x)} y = \underbrace{x}_{q(x)}$$

$\vdots$

$$y(x) = -1 + C \exp\left(\frac{x^2}{2}\right)$$

one arbitrary constant.

<u>Observation:</u>

Recall $$y' \underbrace{-x}_{p(x)} y = \underbrace{x}_{q(x)}$$

<u>Note:</u> The gen. soln. has two parts:

$$y = y_P + y_H$$

y_P = a(ny) particular soln. of the full eqn.

y_H = general soln. of the homog. ODE, i.e. the ODE with $q(x) = 0$.

In our case:

- $y_P = -1$; $y_P' = 0$

  check: $y_P' - x\, y_P = x$

  $$0 - x(-1) \stackrel{?}{=} x \quad \checkmark$$

- $y_H = C \exp\left(\frac{x^2}{2}\right)$

  ~~So~~ is the gen. soln. of homog. ODE

  $$y_H' - x\, y_H = 0$$

  $$\frac{d y_H}{dx}$$

Sep:

$$\int \frac{1}{y_H}\, d y_H = \int x\, dx$$

$$\ln|y_H| = \frac{1}{2}x^2 + A$$

$$y_H = \exp\left(\frac{1}{2}x^2 + A\right) \quad ; \quad C = \exp(A)$$

$$y_H = C\exp\left(\frac{1}{2}x^2\right)$$

Always true even for higher order ODEs.

§3 2nd order ODEs

$$\boxed{\begin{array}{c} F(x, y, y', y'') = 0 \\ + \text{ 2 BC or 2 IC} \end{array}}$$

Linear 2nd order ODEs

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = d(x)$$

If $a(x) \neq 0$

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

Some theory for *linear* 2nd order ODEs

Existence and Uniqueness

Consider the *linear* second-order ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x), \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the functions $p(x)$, $q(x)$ and $r(x)$ are given.

Theorem

> If the functions $p(x)$, $q(x)$ and $r(x)$ are continuous functions of x in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in the entire interval I.

Notes:

- This is the promised extension of the statement for first-order problems. The extension to even higher-order linear ODEs should be obvious...
- If the functions $p(x)$, $q(x)$ and $r(x)$ are "well-behaved" (no jumps, singularities, etc.), the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

The homogeneous ODE & superposition of its solutions

If we set $r(x) = 0$ in the *inhomogeneous* ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x), \qquad \text{(I)}$$

we obtain the corresponding *homogeneous* ODE

$$y'' + p(x)\, y' + q(x)\, y = 0. \qquad \text{(H)}$$

A trivial (?) but useful observation

If $y_1(x)$ and $y_2(x)$ are two solutions of (H) then the linear combination

$$y_3(x) = A\, y_1(x) + B\, y_2(x)$$

is also a solution, for any values of the constants A and B.

Linear independence

To see why this is a useful observation, we need to define the concept of linear independence: Two nonzero functions $y_1(x)$ and $y_2(x)$ are linearly independent if

$$A\, y_1(x) + B\, y_2(x) = 0 \quad \forall x \qquad \Longleftrightarrow \qquad A \equiv B \equiv 0$$

(...just as in linear algebra...).

Examples:

- $y_1(x) = x$ and $y_2(x) = 3\, x^2$ are linearly independent.
- $y_1(x) = x$ and $y_2(x) = 3\, x$ are linearly dependent – they're just multiples of each other.

$$y'' + p y' + q y = 0 \qquad (H)$$

Have two solns: y_1, y_2

Claim: $\boxed{Ay_1 + By_2}$ is also a soln.

Proof: into ODE:

$$\underbrace{Ay_1'' + By_2''}_{y''} + p\underbrace{(Ay_1' + By_2')}_{y'} + q\underbrace{(Ay_1 + By_2)}_{y} \stackrel{?}{=} 0$$

$$A\underbrace{(y_1'' + p y_1' + q y_1)}_{0} +$$

$$B\underbrace{(y_2'' + p y_2' + q y_2)}_{0} \stackrel{?}{=} 0$$

$\square$

lin. independence:

Ex:

$$y_1 = x \qquad y_2 = 3x^2$$

$$A y_1 + B y_2 = 0 \quad \forall x$$

$$\overset{?}{\Longrightarrow} A = B = 0$$

$$A x + B\, 3x^2 \overset{!}{=} 0$$

Can't check for all values of x

Check at:

$x = 1$ $\qquad A + 3B \overset{!}{=} 0$

$x = -1$ $\qquad -A + 3B \overset{!}{=} 0$

$\underline{A = 0}$

$$\Sigma = \quad 6B = 0 \Rightarrow \underline{B = 0}$$

$\Rightarrow$ lin. indep!

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ex: $y_1 = x \qquad y_2 = 3x$

$$A y_1 + B y_2 = 0 \qquad \forall x$$

$$Ax + B\,3x = 0$$

$$x(A + 3B) = 0$$

$$A = -3B \qquad \begin{array}{l} A = -3 \\ B = 1 \end{array}$$

$\Rightarrow$ lin. dep.

Fundamental solutions of the homogeneous ODE

Theorem

Any solution of the homogeneous ODE

$$y'' + p(x)\, y' + q(x)\, y = 0. \qquad \text{(H)}$$

can be written as a linear combination of *any* two non-zero, linearly independent solutions, $y_1(x)$ and $y_2(x)$, say:

$$y(x) = A\, y_1(x) + B\, y_2(x).$$

The two non-zero, linearly independent solutions $\{y_1(x),\, y_2(x)\}$ are called "fundamental solutions" of the homogeneous ODE (H).

Notes:

- The set of fundamental solutions is not unique!

The general solution of the inhomogeneous ODE

Theorem

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + A\, y_1(x) + B\, y_2(x),$$

where:

- A and B are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Notes:

- Note the similarities between the structure of the solution of the linear ODE and the structure of the solution of the linear (algebraic) equation $\mathbf{A}\mathbf{x} = \mathbf{b}$. This is not accidental! There are deep connections between the two fields – matrices and the homogeneous part of a linear ODE are both "linear operators".
- The values of the constants A and B are determined by the boundary or initial conditions.

Example:

$$y'' + \underbrace{\frac{1}{x}}_{p(x)} y' - \underbrace{\frac{1}{x^2}}_{q(x)} y = \underbrace{-\frac{1}{x^2}}_{r(x)}$$

IC: $y(1) = 1$ $\quad X = 1$

$y'(1) = 1$ $\quad Y = 1$

$Z = 1$

$$\left.\begin{array}{l} p(x) = \frac{1}{x} \\ q(x) = -\frac{1}{x^2} \\ r(x) = -\frac{1}{x^2} \end{array}\right\}$$

continuous fcts of x in

$I_1 = \mathbb{R}^-$

$I_2 = \mathbb{R}^+$ (not zero)

$X \in I_2$

$\Rightarrow$ Exists a unique soln. to IVP for $x \in I_2 = \mathbb{R}^+$

- particular soln:

$y_p = 1$

check: $y_p' = y_p'' = 0$
into ODE:

$$0 + 0 - \frac{1}{x^2} 1 \overset{?}{=} -\frac{1}{x^2} \checkmark$$

[y_p not unique:

$y_p = 1 + x$ works too.]

- solns. of the homog. ODE

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0 \qquad (H)$$

$y_1(x) = x \checkmark$

$y_2(x) = \frac{1}{x} \checkmark$

both solve (H); nonzero; lin. indep.

So the general soln of the full eqn is

$$y(x) = \underbrace{1}_{y_p} + A \underbrace{x}_{y_1} + B \underbrace{\frac{1}{x}}_{y_2}$$

Apply ICs:

$$\left.\begin{aligned} y(x=1) &= 1 \\ y'(x=1) &= 1 \end{aligned}\right\} \quad \begin{aligned} A &= \tfrac{1}{2} \\ B &= -\tfrac{1}{2} \end{aligned}$$

$$y(x) = 1 + \frac{1}{2}x - \frac{1}{2x}$$

is the unique soln of IVP.

<u>Note:</u> soln. only exists for $x > 0$.

What happens if we had chosen different fcts for y_p, y_1 & y_2

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E.g.

$$y_p = 1 + x$$ solves full eqn

$$\left.\begin{aligned} y_1 &= x \\ y_2 &= x + \frac{2}{x} \end{aligned}\right\}$$ both solve (H) & are nonzero & lin. indep.

The general soln is:

$$y(x) = \underbrace{1+x}_{y_p} + C\underbrace{x}_{y_1} + D\underbrace{\left(x + \frac{2}{x}\right)}_{y_2}$$

Now apply IC:

$$\left.\begin{aligned} y(x=1) &= 1 \\ y'(x=1) &= 1 \end{aligned}\right\} \quad C = D = -1/4$$

$$y(x) = 1 + x \underbrace{-\frac{1}{4}}_{C} x \underbrace{-\frac{1}{4}}_{D} \left(x + \frac{2}{x}\right)$$

$$y(x) = 1 + \frac{1}{2}x - \frac{1}{2x}$$ Same!

Note: soln. is singular at $x=0$, "as expected" (recall $p(x)$, $q(x)$, $r(x)$ were all singular at $x=0$).

But this is not necessarily the case: for instance, if the ICs are:

$$y(1) = 3/2 \qquad y'(1) = 1/2$$

$$y = 1 + \frac{1}{2} x$$

This is not singular!