Graphical done

\[ \frac{dy}{dx} = f(x, y) \]

Slope!

Ex:

\[ \frac{dy}{dx} = f(x, y) = -\frac{x}{y} \]

\[ \frac{dy}{dx} = -1 \]

\[ \frac{dy}{dx} \rightarrow \pm \infty \]

\[ \frac{dy}{dx} = \pm 1 \]

\[ \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = 0 \]

Supposition: Solution = circular arcs where \( R = \text{const.} \)

\[ y = \pm \sqrt{R^2 - x^2} \]

(true)
Note: On a graph, is a so-called critical point i.e. a point where multiple isolines intersect.

Revisit Example:
Recall: \( f(x, y) = -\frac{x}{y} \) potential problem when \( y = 0 \)

IC \( y(x = x_0) = y \)

Check sketch:

\( y \neq 0 \):
- exactly one curve passes through \((x, y)\)
- but seen only exists for limited range of \( x \)-values

\( y = 0 \):
- can go up & down (non-uniqueness)
- so, only exists for larger or smaller values of \( x \).
3 analytical methods for specific 1st order ODEs

Separable ODEs

1st order ODE is separable if it can be written in the form

\[ g(y) \frac{dy}{dx} = h(x) \]

or

\[ \int g(y(x)) \frac{dy}{dx} dx = \int h(x) dx \]

change of variables

\[ \int g(y) dy = \int h(x) dx + A \]

for an arbitrary constant of integration A
Note: 1) It may not be possible to do the integrals.

2) It may not be possible to solve this for \( f(x) \). In that case we obtain an implicit solution.

Example:

\[
\frac{dy}{dx} = -\frac{x}{y} \quad \text{revisited}
\]

\[
y \frac{dy}{dx} = -x
\]

\[
g(y) \quad h(x)
\]

\[
\int y \, dy = \int -x \, dx
\]

\[
\frac{y^2}{2} + \frac{x^2}{2} = A > 0
\]
\[ x^2 + y^2 = R^2 \quad \text{where} \quad R^2 = 2A \]

\[ y(x) = \pm \sqrt{R^2 - x^2} \]

Apply IC \quad y(x=1) > 0

\[ y(x) = \pm \sqrt{1 - x^2} \]
Example:

\[
\frac{dy}{dx} = e^{x-y} = e^x e^{-y}
\]

\[
e^{\int g(y) \, dy} \frac{dy}{dx} = e^{\int h(x) \, dx}
\]

\[
\int e^{\int g(y) \, dy} \, dy = \int e^x \, dx + A
\]

\[
e^y = e^x + A
\]

\[
y(x) = \ln(e^x + A)
\]

I.C. e.g. \( y(x_0) = y_0 \) (messy)

Alternative way to build in the I.C. definite integral
\[ y(x) = \int_{x_0}^{x} e^{-y} \, dx \]

Should write this as:

\[ y(x) = \int_{x_0}^{x} e^{-y} \, dx \]

\[ e^{-y} = e^{-x} - e^{-x_0} \]

\[ y(x) = e^{-x_0} + e^{-x} - e^{-y} \]
1st order ODEs of homogeneous type

An ODE is of homogeneous type if it can be written as

\[ \frac{dy}{dx} = f\left( \frac{y}{x} \right) \]

**Trick:** Substitute

\[ \frac{y(x)}{x} = z(x) \]
\[ y(x) = x \cdot z(x) \]

\[ \frac{dy}{dx} = \left[ z + x \frac{dz}{dx} \right] = f(z) \]

ODE for z(x) of separable type

\[ \frac{d^2 z}{dx^2} = \frac{f(z) - 2}{x} \]
\[ \int \frac{1}{f(x)} \, dx = \int \frac{dx}{x} \]

DON'T MEMORISE THIS!

**Example:**

\[ x^2 \frac{d^2 y}{dx^2} + y^2 - xy = 0 \]

**Note:** \( y = 0 \) is a solution!

\[ \frac{d^2 y}{dx^2} + \left( \frac{y}{x} \right)^2 - \left( \frac{y}{x} \right) = 0 \]

\[ z = \frac{y}{x} \]

\[ z = \frac{x}{z} \]

\[ y = z \cdot x = z(x) \cdot x \]

\[ \frac{d^2 y}{dx^2} = z + x \frac{d^2 z}{dx^2} \]

\[ x^2 + x \frac{d^2 z}{dx^2} + z^2 = 0 \]
\[ x \frac{d^2 y}{dx^2} = -x^2 \]

\[ \int -\frac{1}{2} \, dx = \int \frac{1}{x} \, dx \]

\[ \frac{1}{2} x = \ln(|x| + c) = \frac{x}{y} \]

\[ y(x) = \frac{x}{\ln(|x| + c)} \]

**Note:** Potential problem at \( x = 0 \).

The other root, \( y = 0 \), is not included in the above family of solutions.
Linear 1st order ODEs

Standard form

\[ y' + p(x)y = q(x) \]

Can be solved by integrating factor:

\[ \mu(x) \text{ s.t. } \mu(x) & \text{ & integrate w.r.t. } x \]

\[ \int (y'\mu + y\mu_1) \, dx = \int \mu q \, dx \]

\[ \text{if } \mu_1 = \mu' \]

then

\[ \int (y'\mu + y'\mu_1) \, dx = \int \frac{d}{dx}(y\mu) \, dx \]

\[ = y\mu \]

Strategy: (i) Choose \( \mu(x) \)
such that
\[ \frac{d\mu}{dx} = \rho(x)\mu \quad (\star) \]

This is a separable ODE for \( \mu(x) \).
Then the integral becomes
\[ \mu g = \int \mu g \, dx \]
\[ g(x) = \frac{1}{\mu(x)} \int \mu g \, dx \]

(\star): \[ \frac{d\mu}{dx} = \rho \mu \]
\[ \int \frac{1}{\mu} \, d\mu = \int \rho \, dx \]
\[ \ln \mu = \int \rho \, dx \]
\[ \mu(x) = \exp\left( \int \rho(x) \, dx \right) \]

Note: const. of integration doesn't matter here.
2) Integrate on $\phi^*$ and multiply \( \phi^* \geq b > \mu(x) \) & integrate

\[
\int (y' + p(x)y) \mu_0 dx = \int q(x) \mu_0 dx
\]

\[
\mu(x) = \frac{1}{\mu} \left[ \int q(x) \mu_0 dx + C \right]
\]

This constant does matter!

Example:

\[
y' + xy = x
\]

\[
p(x) = -x \quad q(x) = x
\]

(1) Integrating factor

\[
\mu(x) = \exp \left( \int p(x) dx \right)
\]
\[ m(x) = \exp\left( -\frac{x^2}{2} \right) \]

\[ = \int x \exp\left( -\frac{x^2}{2} \right) \, dx \]

\[ \int \frac{d}{dx} \left( x \exp\left( -\frac{x^2}{2} \right) \right) \, dx \]

\[ = x \exp\left( -\frac{x^2}{2} \right) \]

\[ \left[ \text{Subst: } z = -\frac{x^2}{2} \right] \]

\[ = -\exp\left( -\frac{x^2}{2} \right) + c \]

\[ g(x) = -1 + c \exp\left( \frac{x^2}{2} \right) \]