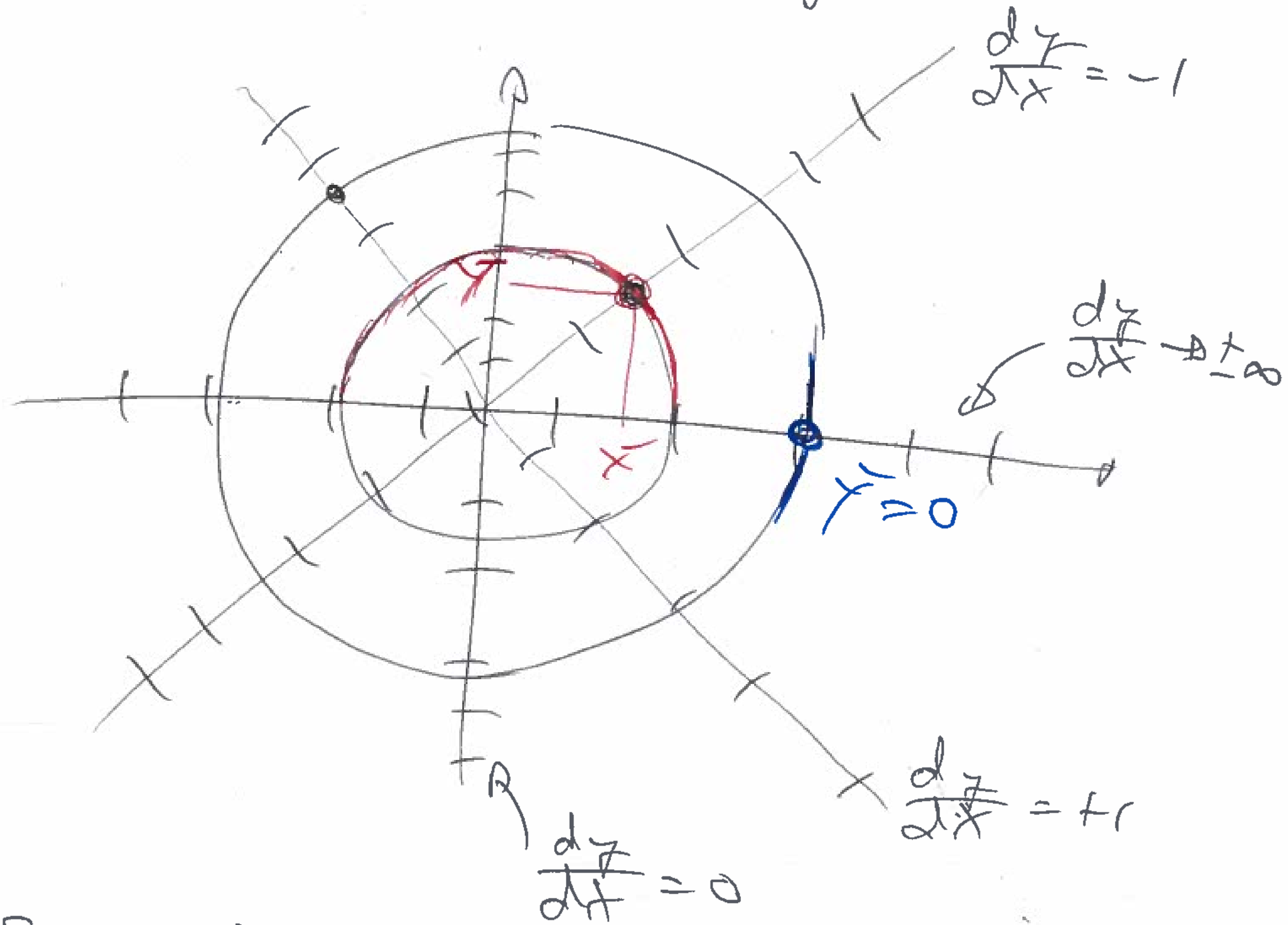
Graphical sol'ns

$$\frac{dy}{dx} = f(x, y)$$

slope!

Ex: $\frac{dy}{dx} = f(x,y) = -\frac{x}{y} = s$



Suggestion: solution = circular arcs

$y = \pm\sqrt{R^2 - x^2}$ where R = const.

(true).

Note: origin is a so-called "critical point" i.e. a point where multiple isoclines intersect.

Revisit E & U:

recall: $f(x,y) = -\frac{x}{y}$

IC $y(x=X) = Y$

potential problem when $Y = 0$

check sketch:

$Y \neq 0$:
- exactly one curve passes through (X, Y)
- but soln only exists for limited range of x-values

$Y = 0$:
- can go up & down (non-uniqueness)
- soln. only exists for larger or smaller values of x.

3 analytical methods for specific 1st order ODEs

~~Spe~~

Separable ODEs

1st order ODE is separable if it can be written in the form

$$g(y)\frac{dy}{dx} = h(x)$$

$$\int g(y(x))\frac{dy}{dx}dx = \int h(x)\,dx$$

change of variables

$$\int g(y)\,dy = \int h(x)\,dx + A$$

for an arbitrary constant of integration A

Note: 1) It may not be possible to do the integrals

2) It may not be possible to solve this for $y(x)$. In that case we obtain an implicit soln.

Example:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{revisited}$$

$$y(x=1) = 0$$

$$\underbrace{y}_{g(y)} \frac{dy}{dx} = \underbrace{-x}_{h(x)}$$

$$\int y \, dy = \int -x \, dx$$

$$\frac{y^2}{2} + \frac{x^2}{2} = A > 0$$

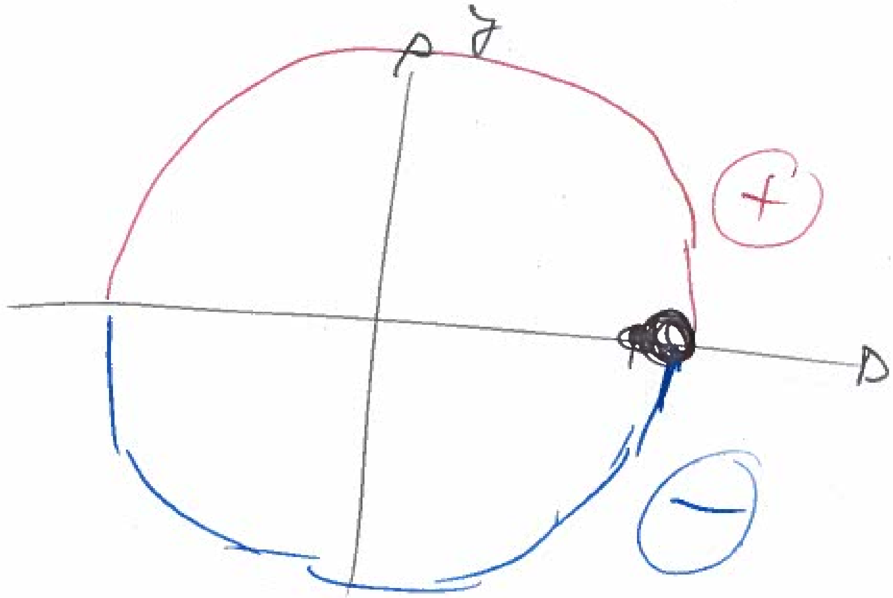
5

$x^2+y^2=R^2$ where $R^2=2A$

$y(x)=\pm\sqrt{R^2-x^2}$

Apply IC $y(x=1)=0$

$y(x)=\pm\sqrt{1-x^2}$



Example:

$$\frac{dy}{dx} = e^{x-y} = e^{x} e^{-y}$$

$$\underbrace{e^{y}}_{g(y)} \frac{dy}{dx} = \underbrace{e^{x}}_{h(x)}$$

$$\int e^{y}\,dy = \int e^{x}\,dx + A$$

$$e^{y} = e^{x} + A$$

$$y(x) = \ln(e^{x} + A)$$

IC: e.g.: $y(x_0) = y_0$ (messy)

Alternative way to build in the IC. Definite integral

$$\int_{y_0}^{y(x)} e^{y}\,dy = \int_{x_0}^{x} e^{x}\,dx$$

Should write this as:

$$\int_{y_0}^{y(x)} e^{\eta}\,d\eta = \int_{x_0}^{x} e^{\xi}\,d\xi$$

$$e^{y(x)} - e^{y_0} = e^{x} - e^{x_0}$$

$$y(x) = \ln\left(e^{x} - e^{x_0} + e^{y_0}\right)$$

1st order ODEs of homogeneous type

ODE is of homog. type if it can be written as

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$

Trick: Substitute

$$\frac{y(x)}{x} = z(x)$$

$$y(x) = x\, z(x)$$

$$\boxed{\frac{dy}{dx} =} \boxed{z + x\frac{dz}{dx} = f(z)}$$

ODE for $z(x)$ of separable type

$$\frac{dz}{dx} = \frac{f(z) - z}{x}$$

$$\int \frac{1}{f(z)-z}\,dz = \int \frac{dx}{x}$$

DON'T MEMORISE THIS!

Example:

$$x^2 \frac{dy}{dx} + y^2 - xy = 0 \quad \Big| \cdot \frac{1}{x^2}$$

$x \neq 0$

Note: $y = 0$ is a soln!

$$\frac{dy}{dx} + \underbrace{\left(\frac{y}{x}\right)^2}_{z^2} - \underbrace{\left(\frac{y}{x}\right)}_{z} = 0$$

$$z = \frac{y}{x}$$

$$y = zx = z(x)\,x$$

$$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\cancel{z} + x\frac{dz}{dx} + z^2 - \cancel{z} = 0$$

$$x \frac{dz}{dx} = -z^2$$

$$\int -\frac{1}{z^2}\,dz = \int \frac{1}{x}\,dx$$

$$\frac{1}{z} = \boxed{\ln|x| + C = \frac{x}{y}}$$

$$y(x) = \frac{x}{\ln|x| + C}$$

Note:
- potential problem at $x = 0$
- The other soln, $y = 0$, is not included in the above family of solns.

Linear 1st order ODEs

Standard form

$$y' + p(x)\,y = q(x)$$

Can be solved by integrating factor:

multiply ODE by some fct. $\mu(x)$ & integrate w.r.t. x

$$\int (y'\mu + y\,p\mu)\,dx = \int \mu q\,dx$$

if $p\mu = \mu'$

then

$$\int (y'\mu + y\mu')\,dx = \int \frac{d}{dx}(y\mu)\,dx$$

$$= y\mu$$

Strategy: (1) Choose $\mu(x)$

such that

$$\frac{d\mu}{dx} = p(x)\mu \qquad (*)$$

This is a separable ODE for $\mu(x)$.

Then the integral becomes

$$\mu y = \int \mu q \, dx$$

$$y(x) = \frac{1}{\mu(x)} \int \mu q \, dx$$

$(*)$: $\frac{d\mu}{dx} = p\mu$

$$\int \frac{1}{\mu} d\mu = \int p \, dx$$

$$\ln \mu = \int p \, dx$$

$$\mu(x) = \exp\left(\int p(x) \, dx\right)$$

<u>Note:</u> const. of integration doesn't matter here.

2. ~~Integrate ODE~~
Multiply ODE by $\mu(x)$
& integrate

$$\underbrace{\int (y' + py)\mu \, dx}_{\mu y} = \int q\mu \, dx$$

$$\boxed{y(x) = \frac{1}{\mu}\left[\int q\mu \, dx + c\right]}$$

This constant does matter!

Example:

$$y' \underbrace{-x}y = \underbrace{x}$$

$p(x) = -x$ $q(x) = x$

1. integrating factor

$$\mu(x) = \exp\left(\int p(x) \, dx\right)$$

(14

$$\mu(x) = \exp\left(-\frac{x^2}{2}\right)$$

2

$$\int \left(y' \exp\left(-\frac{x^2}{2}\right) - xy \exp\left(-\frac{x^2}{2}\right) \right) dx$$

$$= \int x \exp\left(-\frac{x^2}{2}\right) dx$$

$$\underline{\int \frac{d}{dx}\left(y \exp\left(-\frac{x^2}{2}\right)\right)} dx$$

$$y(x) \exp\left(-\frac{x^2}{2}\right) = \int x \exp\left(-\frac{x^2}{2}\right) dx$$

[Subst: $z = -\frac{x^2}{2}$]

$$= -\exp\left(-\frac{x^2}{2}\right) + C$$

$$\underline{\underline{y(x) = -1 + C \exp\left(\frac{x^2}{2}\right)}}$$