m (mass)

$f(t) \longmapsto x(t)$

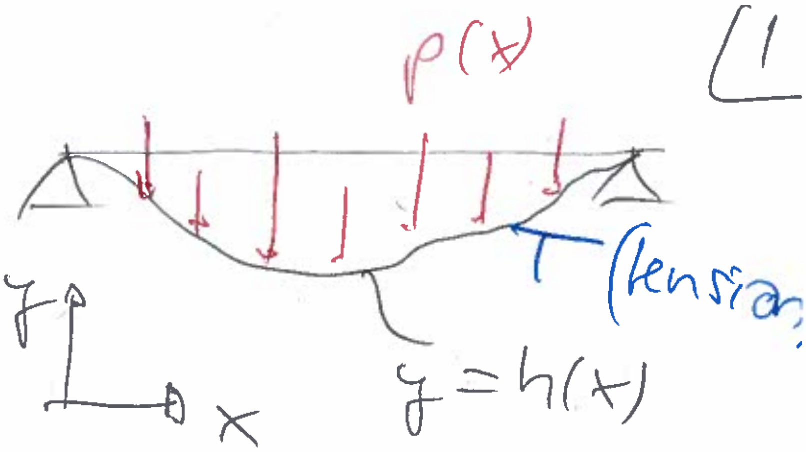
$$m \frac{d^2 x}{dt^2} = f(t)$$

IC: $x(t=0) = x_0$

$$\left.\frac{dx}{dt}\right|_{t=0} v_0$$

E&U : "obvious"

$$x(t) = \frac{1}{2}\frac{F_0}{m}t^2 + v_0 t + x_0$$



$$T \frac{d^2 h}{dx^2} = \rho(x)$$

BC: $h(x=0) = 0$

$h(x=L) = L$

E&U: "obvious"

$$h(x) = \frac{1}{2}\frac{\rho_0}{T}(x^2 - Lx)$$

Finally a counterexample for uniqueness:

$$y' = y^{1/2} \qquad y(x=0) = 0$$

one IC for 1^{st} order ODE

Spot a soln:

$$y(x) \equiv 0$$

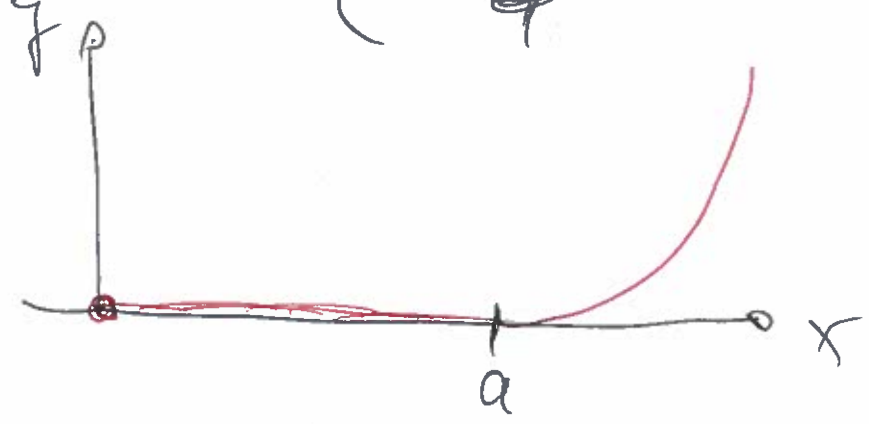
and

$$y(x) = \frac{1}{4} x^2$$

both solve ODE & satisfy IC.

In fact:

$$y(x) = \begin{cases} 0 & \text{for } 0 \le x \le a \\ \frac{1}{4}(x-a)^2 & \text{for } x > a \end{cases}$$

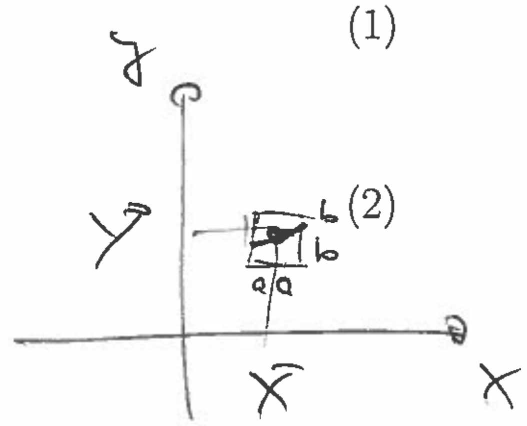


for any constant a.

Existence and uniqueness theorem for 1st order ODEs

Consider the first-order ODE in its explicit form

$$\frac{dy}{dx} = f(x,y), \tag{1}$$



subject to the initial condition

$$y(X) = Y, \tag{2}$$

where the constants X and Y are given.

Theorem

> If $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous functions of x and y in a region $\cancel{0<}|x-X| < a$ and $\cancel{0<}|y-Y| < b$, where $a, b > 0$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $\cancel{0<}|x-X| < h \leq a$. where $h > 0$

Notes:

- The statement is easily generalised to higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

A pretty weak statement then....

Examples:

$$y' = \underbrace{\sin(xy)}_{f(x,y)} \qquad y(x=0) = 1$$

$$X = 0$$
$$Y = 1$$

$$\left.\begin{aligned} f(x,y) &= \sin(xy) \\ \frac{\partial f}{\partial y} &= x\cos(xy) \end{aligned}\right\}$$ both are continuous fcts of x & y everywhere, specifically at $x = X$ & $y = Y$

$\Rightarrow$ unique soln exists <u>in vicinity</u> of $x = X = 0$.

Ex: $y' = \underbrace{y^2}_{f(x,y)}$ $\qquad y(0) = 1$

$$\left.\begin{aligned} f(x,y) &= y^2 \\ \frac{\partial f}{\partial y} &= 2y \end{aligned}\right\}$$ continuous fcts of x & y everywhere.

$\Rightarrow$ unique soln exists in the vicinity of $x = 0$.

In fact:

$$y(x) = \frac{1}{1-x}$$

is that unique soln.

Note that the soln has a singularity at $x = 1$.

Existence and uniqueness theorem for *linear* 1st order ODEs

Consider the *linear* first-order ODE

$$\frac{dy}{dx} + p(x)\,y = q(x), \tag{3}$$

subject to the initial condition

$$y(X) = Y, \tag{4}$$

where the constants X and Y and the functions $p(x)$ and $q(x)$ are given.

Theorem

> If the functions $p(x)$ and $q(x)$ are continuous functions in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (3) and (4) in the entire interval I.

Notes:

- The statement is again easily generalised to higher-order ODEs.
- The theorem provides a "much more global" statement. In fact, if the functions $p(x)$ and $q(x)$ are "well-behaved" (no jumps, singularities, etc.) the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

This is a much stronger statement and explains in part why (some) mathematicians love (only) linear problems.

Examples:

$$\underbrace{}_{}y' + \underbrace{x}_{p(x)} y = \underbrace{x}_{q(x)} \qquad y(x=0) = 2$$

$\bar{X} = 0$

$\bar{Y} = 2$

$$\left.\begin{aligned} p(x) &= x \\ q(x) &= x \end{aligned}\right\}$$ cont. fcts of x everywhere, in $\underline{I} = \mathbb{R}$ where $\bar{X} \in I$

$\Rightarrow$ unique soln. exists for $x \in \underline{I} = \mathbb{R}$.

In fact:

$$y(x) = 1 + \exp\left(-\frac{1}{2}x^2\right)$$

is that soln.

Ex: $\underbrace{y' + \frac{1}{x}}_{p(x)} y = \underbrace{2}_{q(x)}$

$\left.\begin{array}{l} p(x) = \frac{1}{x} \\ q(x) = 2 \end{array}\right\}$ continuous fcts of x in two intervals:

$I_1 = (-\infty, 0)$

$I_2 = (0, \infty)$

$\Rightarrow$ unique soln exists in either of these intervals.

In fact, the general soln is

$$y(x) = x + \frac{A}{x}$$

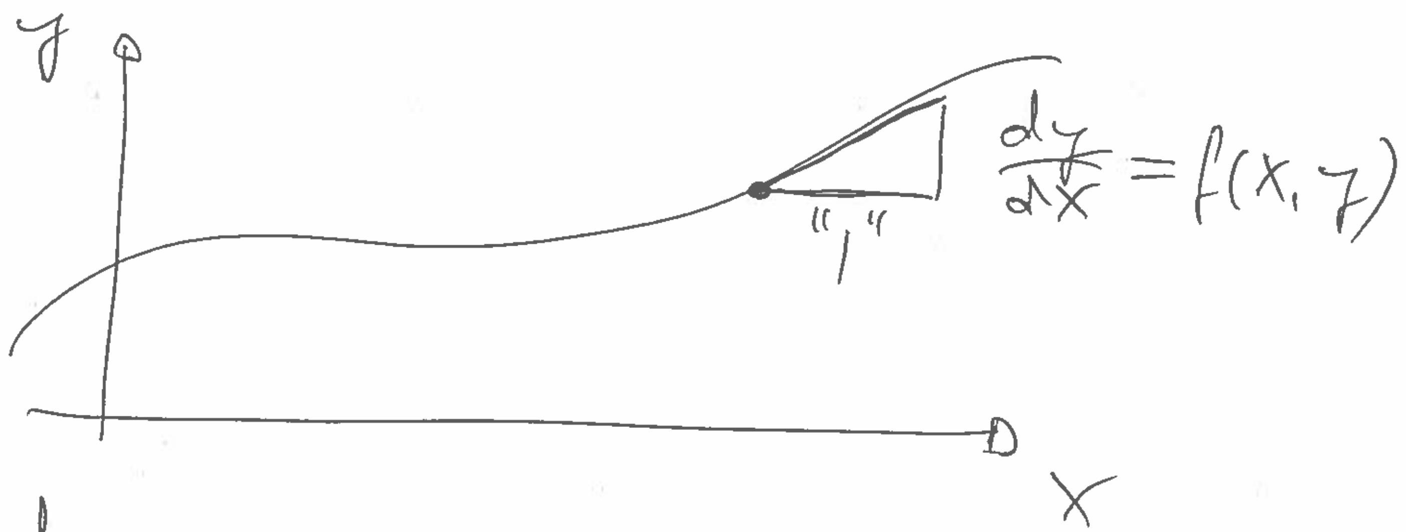
for an arbitrary constant A, determined by IC.

- Soln. exists in I_1 & I_2
- Soln. may not exist at $x = 0$.

§2 First order ODEs

$$y' = f(x, y)$$

I Graphical approach



$\frac{dy}{dx}$ is the slope of $y(x)$.

$\Rightarrow$ $f(x, y)$ defines the slope of the soln. curve(s)!

<u>Def</u>: The <u>direction field</u> of the ODE $\frac{dy}{dx} = f(x, y)$ is the set of all vectors that have the same direction as $\underline{d} = \begin{pmatrix} 1 \\ \frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} 1 \\ f(x, y) \end{pmatrix}$

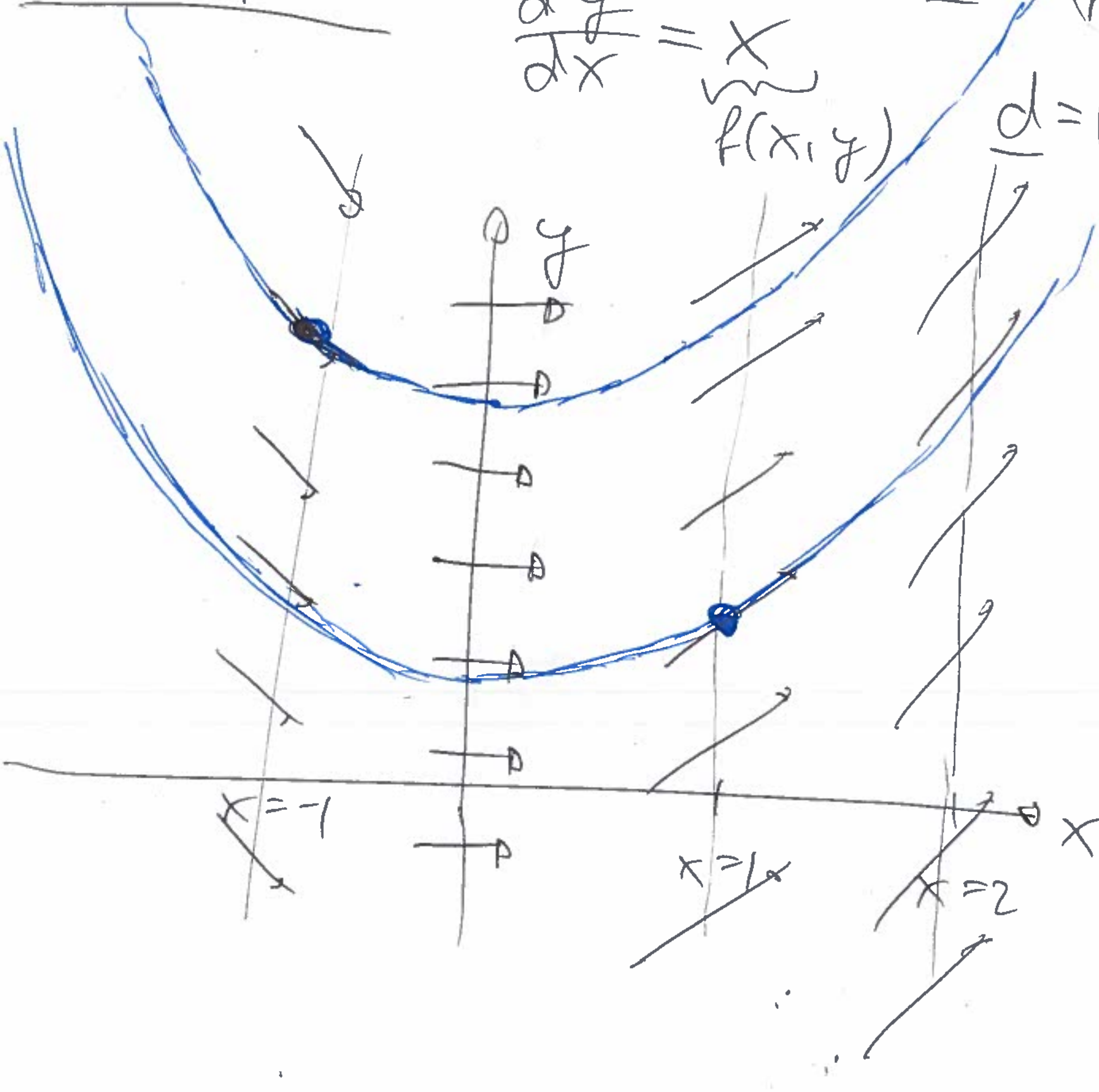
Def: ~~The~~ Integral curves are curves that are everywhere tangent to the direction field. They represent solns. of the ODE.

Example:

$$\frac{dy}{dx} = \underbrace{x}_{f(x,y)}$$

$$\underline{d} = \begin{pmatrix} 1 \\ f(x,y) \end{pmatrix}$$

$$\underline{d} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

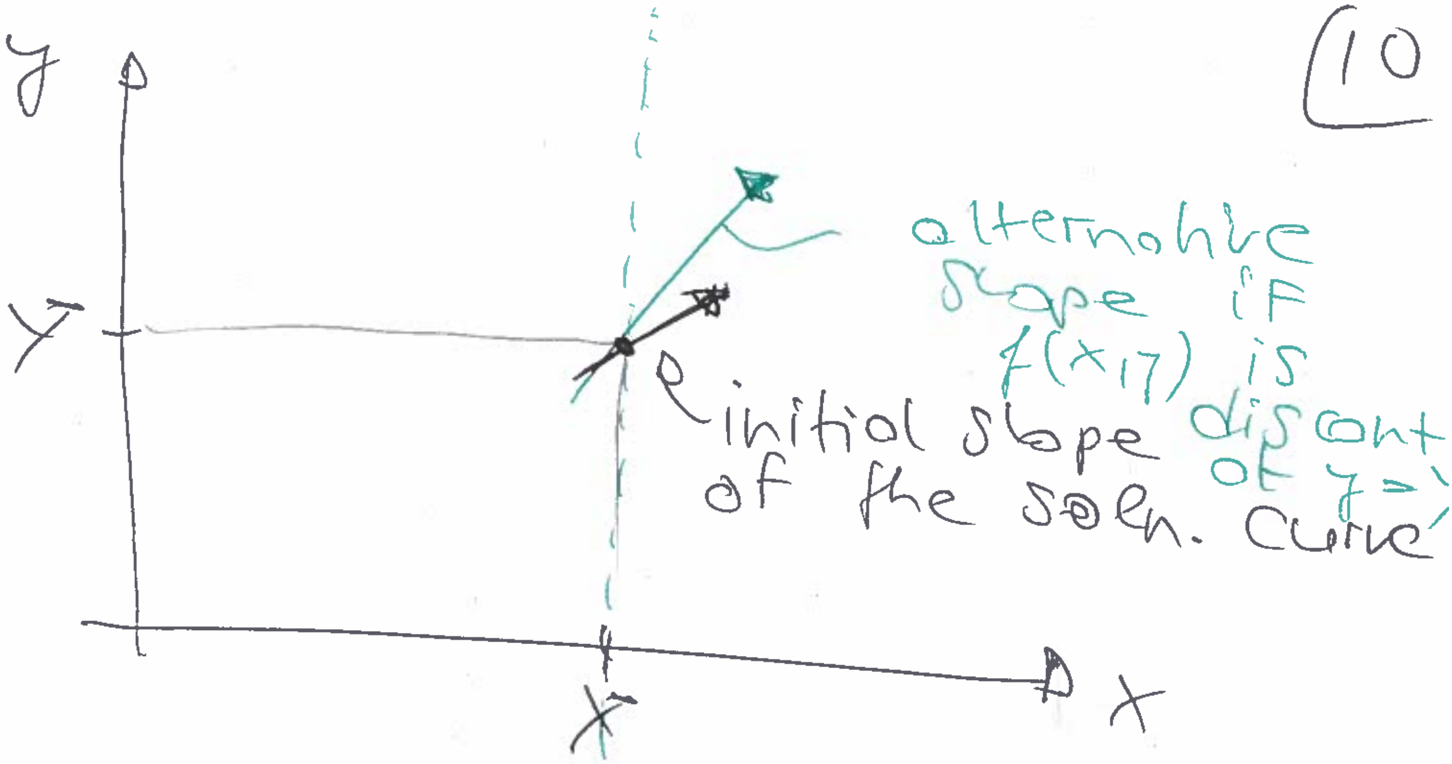


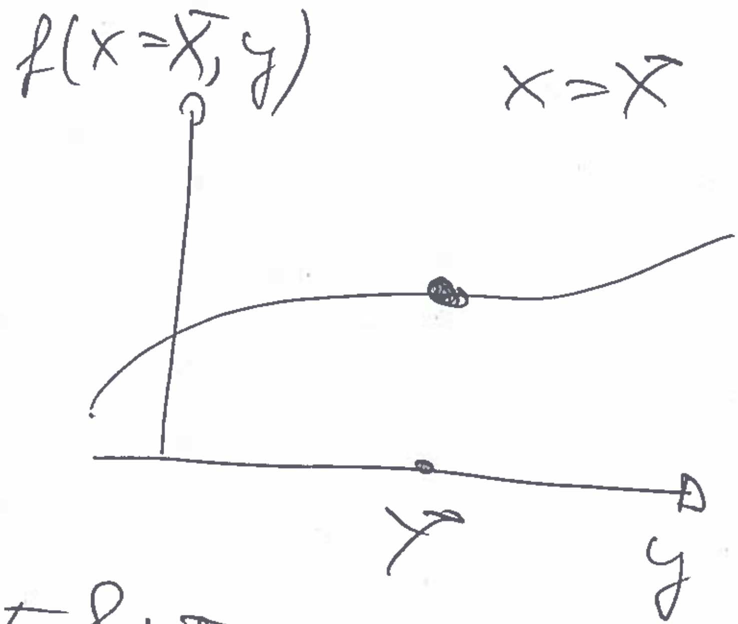
In fact, the general soln is

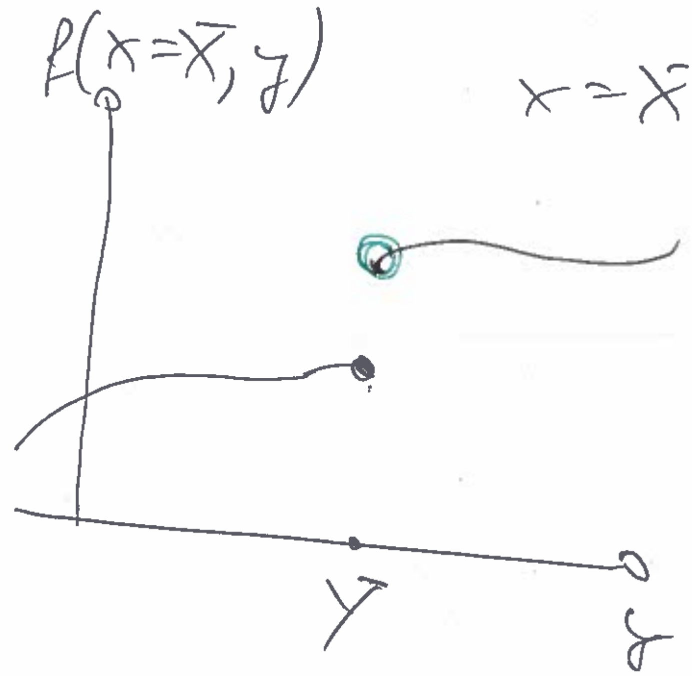
$$y(x) = \frac{1}{2}x^2 + C$$

Graphical soln already illustrates the character of the soln & E&U.

In fact, how can the soln. starting from an IC ever be not unique?







E&U: unique!

not unique

Observation:

When sketching soln. curves, it is helpful to identify so-called <u>isoclines</u> = lines in x-y plane where $\frac{dy}{dx}$ of the soln is constant.

Example:

$$y' = \frac{dy}{dx} = \underbrace{-\frac{x}{y}}_{\text{slope, } s\text{, of the soln. curve.}} = f(x,y)$$

$\boxed{-\frac{x}{y} = s}$ is the eqn. for all points where the soln has slope s.