

m (mass)



$$F(t) \rightarrow x(t)$$

$$m \frac{d^2 x}{dt^2} = F(t)$$

IC: $x(t=0) = x_0$

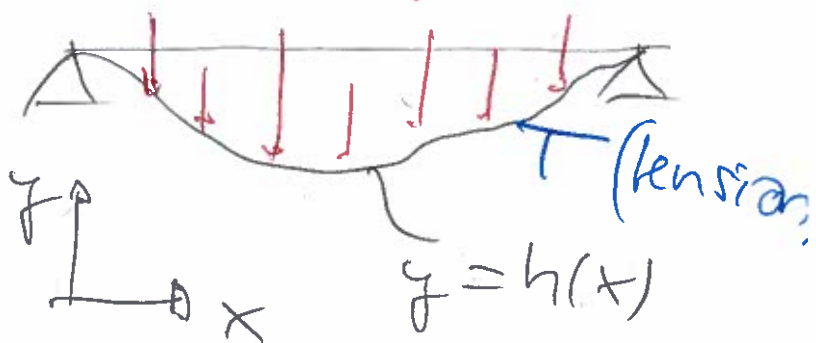
$$\left. \frac{dx}{dt} \right|_{t=0} = v_0$$

E&U: "obvious" \geq &U: "obvious"

$$x(t) = \frac{1}{2} \frac{F_0}{m} t^2 + v_0 t + x_0$$

$\rho(x)$

L



$$T \frac{d^2 h}{dx^2} = \rho(x)$$


BC: $h(x=0) = 0$

$$h(x=L) = L$$

$$h(x) = \frac{1}{2} \frac{\rho_0}{T} (x^2 - Lx)$$

finally a counter example \hookrightarrow
for uniqueness:

$$y' = y^{1/2} \quad y(x=a) = 0$$

one IC for 1st order ODE


Spot a soln:

$$f(x) \equiv 0$$

and

$$f(x) = \frac{1}{4} x^2$$

both solve ODE & satisfy IC.

In fact:

$$f(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \frac{1}{4} (x-a)^2 & \text{for } x > a \end{cases}$$



for any
constant
 a .

Existence and uniqueness theorem for 1st order ODEs

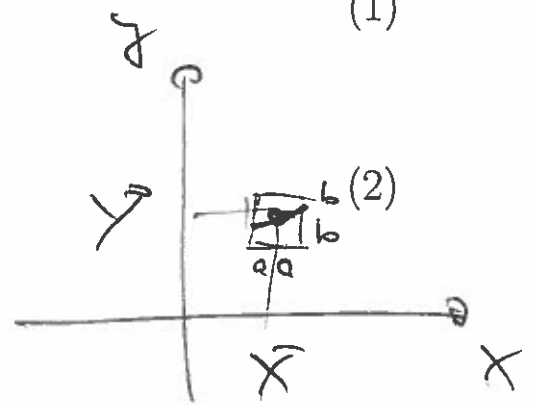
Consider the first-order ODE in its explicit form

$$\frac{dy}{dx} = f(x, y), \quad (1)$$

subject to the initial condition

$$y(X) = Y,$$

where the constants X and Y are given.



Theorem

If $f(x, y)$ and $\frac{\partial f(x, y)}{\partial y}$ are continuous functions of x and y in a region $0 < |x - X| < a$ and $0 < |y - Y| < b$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

Notes:

- The statement is easily generalised to higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! \implies An IVP may still have a unique solution even if the conditions are violated.

A pretty weak statement then....

Examples:

3

$$y' = \underbrace{\sin(xy)}_{f(x,y)}$$

$$y(x=0) = 1$$



$$x = 0$$

$$y = 1$$

$$\left. \begin{aligned} f(x,y) &= \sin(xy) \\ \frac{\partial f}{\partial y} &= x \cos(xy) \end{aligned} \right\}$$

both are continuous fcts of x & y everywhere, specifically at $x = \bar{x}$ & $y = \bar{y}$

⇒ unique soln exists in vicinity of $x = \bar{x} = 0$.

Ex: $y' = \underbrace{y^2}_{f(x,y)}$

$y(0) = 1$

(4)

$$\left. \begin{aligned} f(x,y) &= y^2 \\ \frac{\partial f}{\partial y} &= 2y \end{aligned} \right\}$$

continuous
fcts of x & y
everywhere.

\Rightarrow unique soln exists in
the vicinity of $x=0$.

In fact:

$$y(x) = \frac{1}{1-x}$$

is that unique soln.

Note that the soln has
a singularity at $x=1$.

Existence and uniqueness theorem for *linear* 1st order ODEs

Consider the *linear* first-order ODE

$$\frac{dy}{dx} + p(x)y = q(x), \quad (3)$$

subject to the initial condition

$$y(X) = Y, \quad (4)$$

where the constants X and Y and the functions $p(x)$ and $q(x)$ are given.

Theorem

If the functions $p(x)$ and $q(x)$ are continuous functions in an interval I , and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (3) and (4) in the entire interval I .

Notes:

- The statement is again easily generalised to higher-order ODEs.
- The theorem provides a “much more global” statement. In fact, if the functions $p(x)$ and $q(x)$ are “well-behaved” (no jumps, singularities, etc.) the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

This is a much stronger statement and explains in part why (some) mathematicians love (only) linear problems.

Examples:

(5)

$$y' + \underbrace{x}_p(x)y = \underbrace{x}_q(x) \quad y(x=0) = 2$$
$$\bar{x} = 0$$
$$\bar{y} = 2$$

$$\left. \begin{aligned} p(x) &= x \\ q(x) &= x \end{aligned} \right\}$$

cont. fcts of x everywhere,
in $I = \mathbb{R}$
where $x \in I$

\Rightarrow unique soln. exists for
 $x \in I = \mathbb{R}$.

In fact:

$$y(x) = 1 + \exp\left(-\frac{1}{2}x^2\right)$$

is that soln.

Ex: $y' + \underbrace{\frac{1}{x}}_{p(x)} y = \underbrace{2}_{q(x)}$

$p(x) = \frac{1}{x}$ } continuous fcts of x in two intervals:
 $q(x) = 2$ }

$$I_1 = (-\infty, 0)$$

$$I_2 = (0, \infty)$$

\Rightarrow unique soln exists in either of these intervals.

In fact, the general soln is

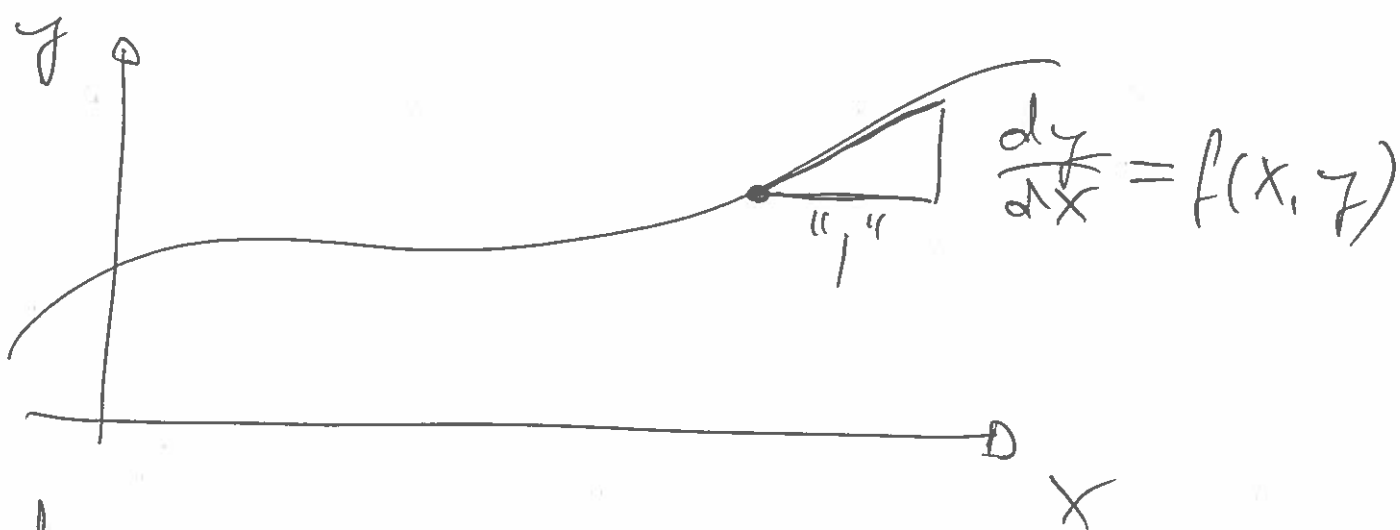
$$y(x) = x + \frac{A}{x} \quad \text{for an arbitrary constant } A, \text{ determined by IC.}$$

- Soln. exists in I_1 & I_2
- Soln. may not exist at $x = 0$.

§ 2 First order ODEs [7]

$$y' = f(x, y)$$

I Graphical approach



$\frac{dy}{dx}$ is the slope of $y(x)$.

$\Rightarrow f(x, y)$ defines the slope of the soln. curve(s)!

Def: The direction field of the ODE $\frac{dy}{dx} = f(x, y)$ is the set of all vectors that have the same direction as $\underline{d} = \left(\frac{dx}{dy} \right) = \left(f(x, y) \right)$

Def: ~~That~~ Integral curves (P)

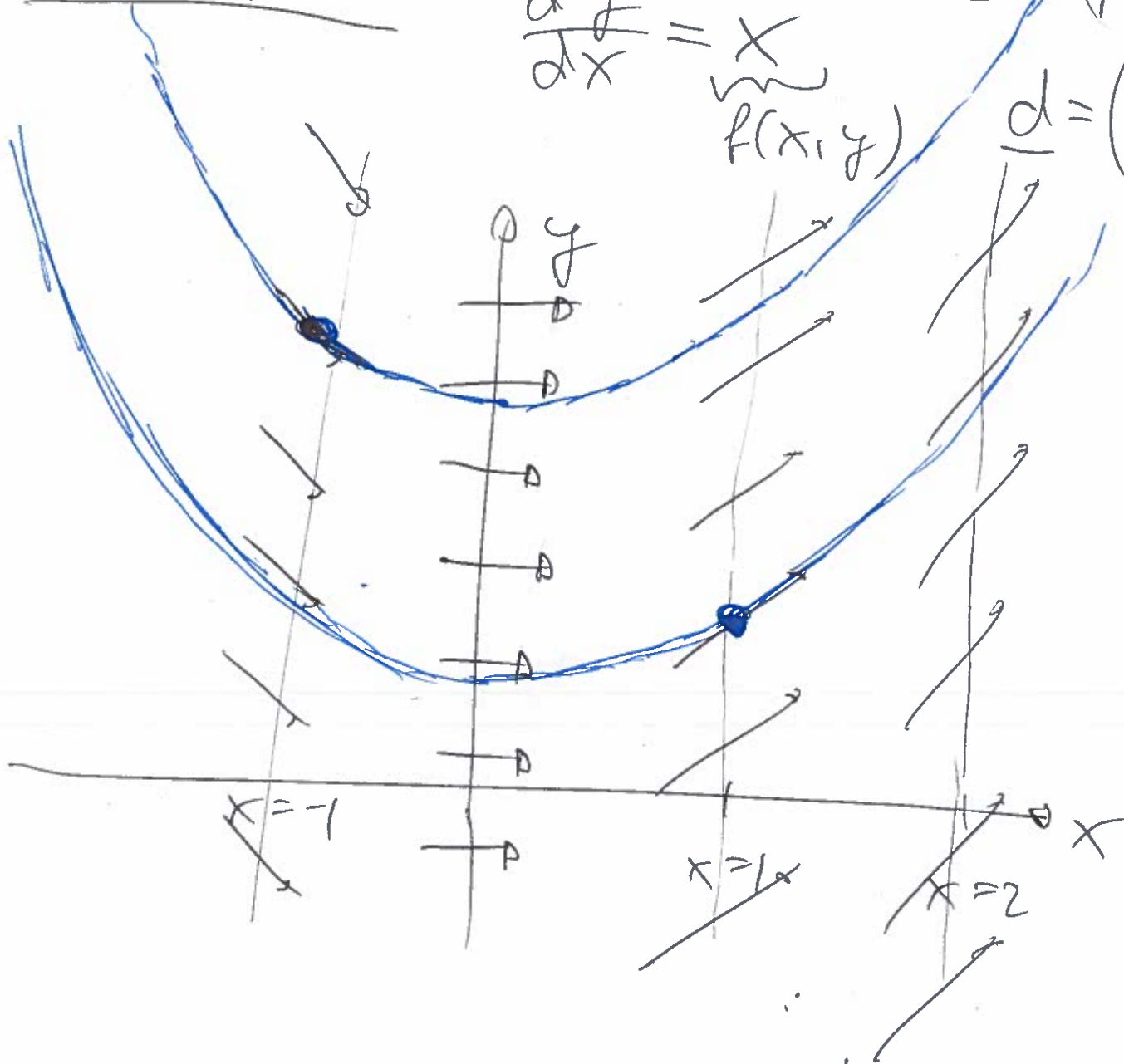
are curves that are everywhere tangent to the direction field.

They represent solns. of the ODE.

Example:

$$\frac{dy}{dx} = \underbrace{x}_{f(x,y)}$$

$$\underline{d} = \begin{pmatrix} 1 \\ f(x,y) \end{pmatrix}$$
$$\underline{d} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

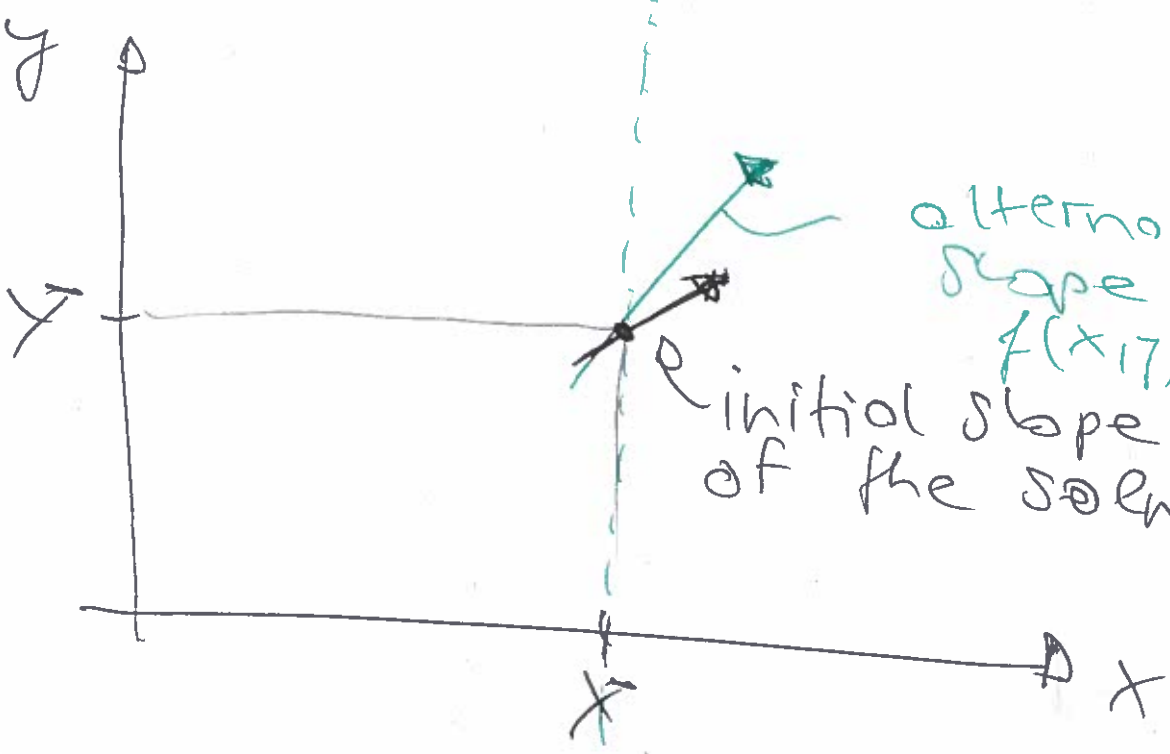


In fact, the general soln (9)
is

$$y(x) = \frac{1}{2}x^2 + C$$

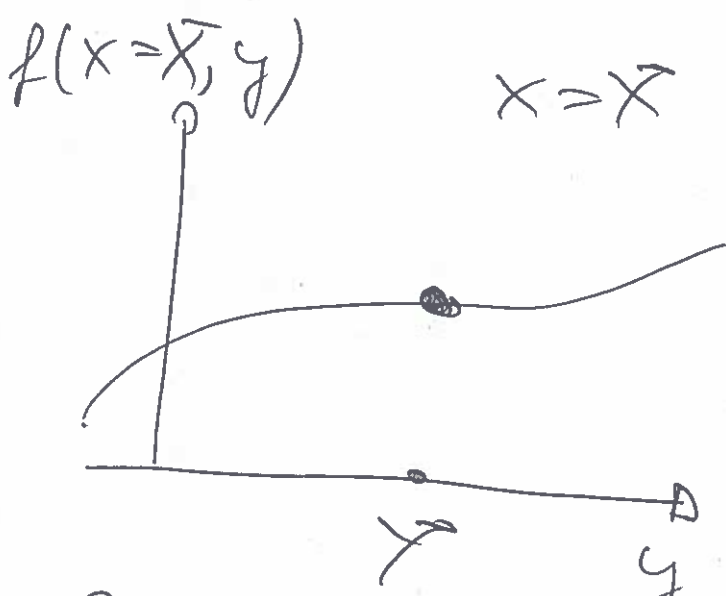
Graphical soln already
illustrates the character
of the soln & E&U.

In fact, how can the soln.
Starting from an IC ever
be not unique?

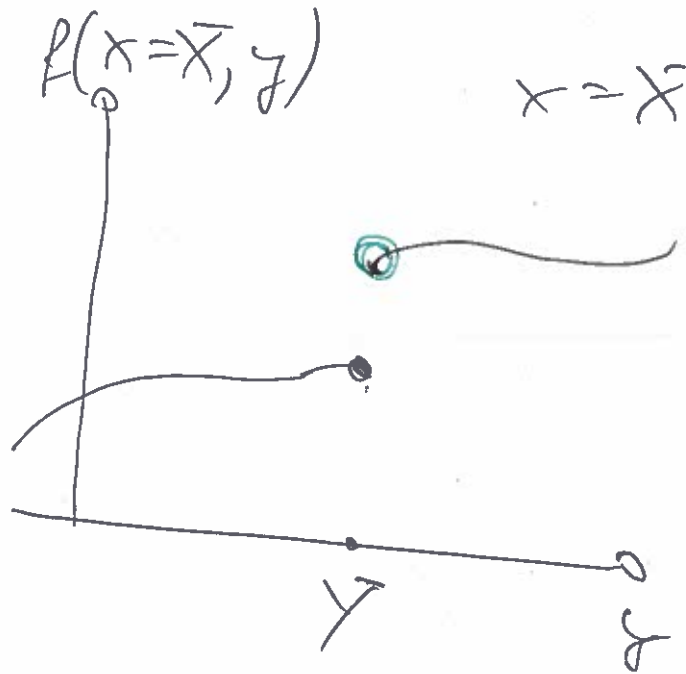


alternative slope if $f(x|y)$ is discontinuous at $y=...$

initial slope of the soln. curve



E&U: unique!



not unique

Observation:

(11)

When sketching soln. curves, it is helpful to identify so-called isoclines = lines in x - y plane where $\frac{dy}{dx}$ of the soln is constant.

Example:

$$y' = \frac{dy}{dx} = -\frac{x}{y} = f(x, y)$$

slope, f' , of the soln. curve.

$$-\frac{x}{y} = f'$$

is the eqn. for all points where the soln has slope f' .

