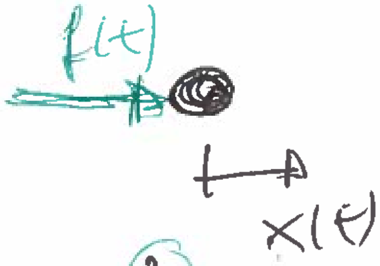


$$m\frac{d^2x}{dt^2} = f(t)$$

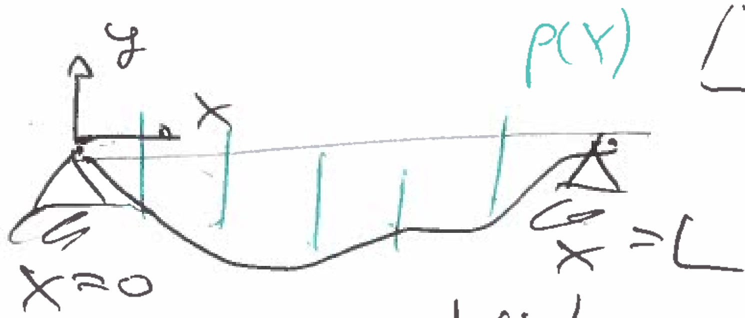
2nd order ODE

$x(t=0) = x_0$

$\left.\frac{dx}{dt}\right|_{t=0} = v_0$

IVP

E & U "of course"



string eq: $y = h(x)$

$$T\frac{d^2h}{dx^2} = p(x)$$

2nd order ODE

$h(x=0) = 0$

$h(x=L) = 0$

BVP

E & U "of course"

Example:

$p(x) = p_0 = \text{const}$

$$h(x) = \frac{1}{2}\frac{p_0}{T}(x^2 - Lx)$$

(EXERCISE)

NOTE: Mathematically the ODEs are exactly the same. "Application" decides if we need ICs or BCs.

Counterexample for E&U

$$y' = y^{1/2} \qquad y(0) = 0$$

one IC for 1st order ODE

Spot two solns:

$$y \equiv 0$$

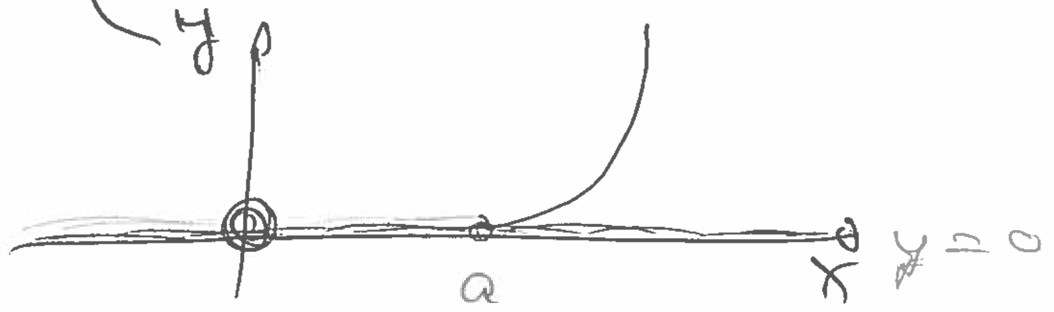
and $y = \frac{1}{4}x^2$

$$y' = \frac{1}{2}x \; ; \; \sqrt{y} = \frac{1}{2}x \quad \checkmark$$

$$y(0) = 0 \quad \checkmark$$

In fact:

$$y = \begin{cases} 0 & \text{for } 0 \le x \le a \\ \frac{1}{4}(x-a)^2 & \text{for } x > a \end{cases}$$



Existence and uniqueness theorem for 1st order ODEs

Consider the first-order ODE in its explicit form

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

subject to the initial condition

$$y(X) = Y, \tag{2}$$

where the constants X and Y are given.

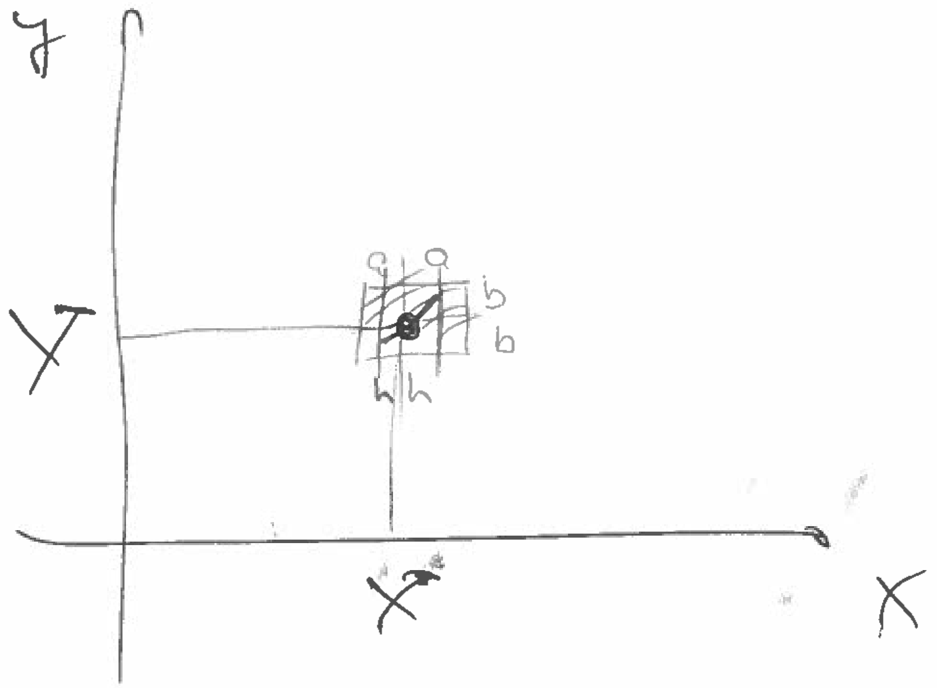
Theorem

If $f(x, y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous functions of x and y in a region $0 < |x - X| < a$ and $0 < |y - Y| < b$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

Notes:

- The statement is easily generalised to higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

A pretty weak statement then....



Examples:

① $y' = \underbrace{\sin(xy)}_{f(x,y)}$ $\qquad y(0) = 1$

$\bar{X} = 0$

$\bar{Y} = 1$

$$\left.\begin{aligned} f(x,y) &= \sin(xy) \\ \frac{\partial f}{\partial y} &= x\cos(xy) \end{aligned}\right\}$$ both continuous fcts of x & y for all values of x & y

$\Rightarrow$ unique soln exists in the vicinity of $x = 0$.

[Note: Eqn cannot be solved in closed form.]

② $y' = \underbrace{y^2}_{f(x,y)}$ $y(0) = 1$

$\bar{x} = 0$

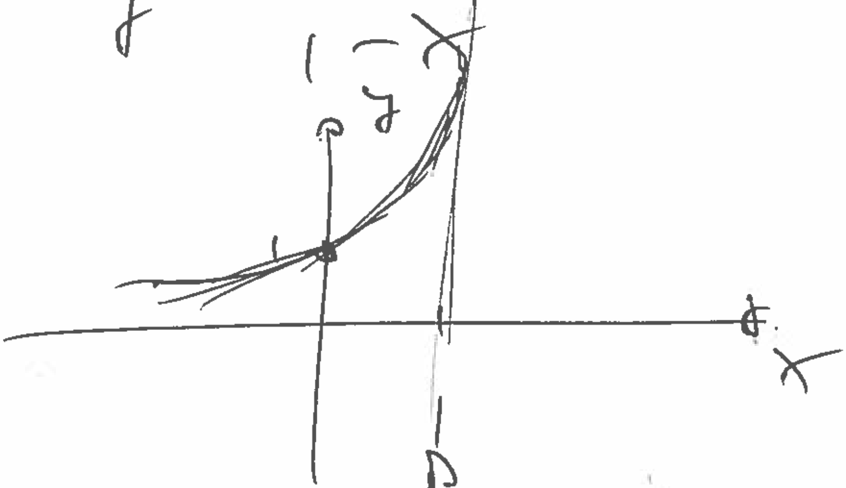
$\bar{y} = 1$

$$\left.\begin{aligned} f(x,y) &= y^2 \\ \frac{\partial f}{\partial y} &= 2y \end{aligned}\right\} \text{ both are continuous for all } x \,\&\, y$$

$\Rightarrow$ unique soln must exist in vicinity of $x = 0$.

In fact:

$y = \dfrac{1}{1-x}$ is that soln.



soln only exists for $x < 1$.

Existence and uniqueness theorem for *linear* 1st order ODEs

Consider the *linear* first-order ODE

$$\frac{dy}{dx} + p(x)\, y = q(x), \tag{3}$$

subject to the initial condition

$$y(X) = Y, \tag{4}$$

where the constants X and Y and the functions $p(x)$ and $q(x)$ are given.

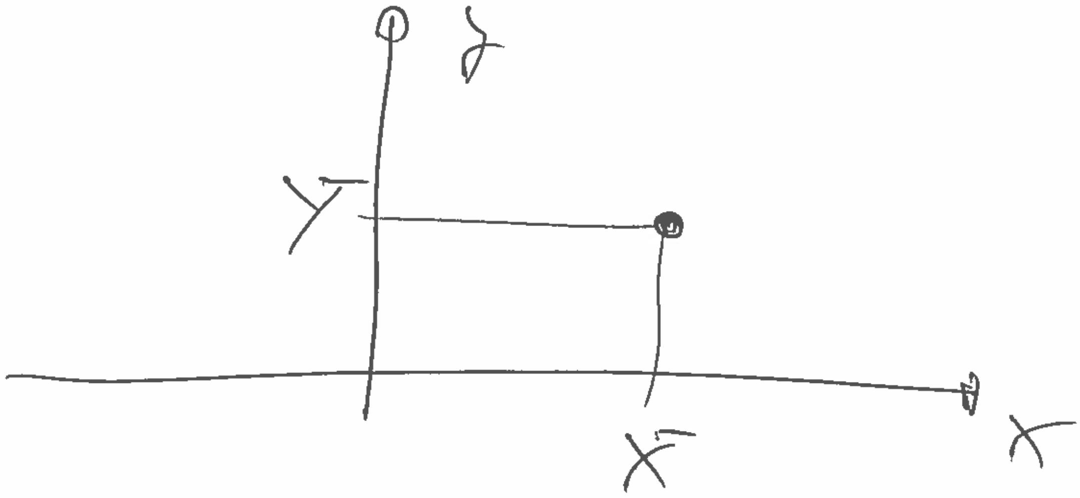
Theorem

> If the functions $p(x)$ and $q(x)$ are continuous functions in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (3) and (4) in the entire interval I.

Notes:

- The statement is again easily generalised to higher-order ODEs.
- The theorem provides a "much more global" statement. In fact, if the functions $p(x)$ and $q(x)$ are "well-behaved" (no jumps, singularities, etc.) the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

This is a much stronger statement and explains in part why (some) mathematicians love (only) linear problems.



Examples:

① $\underset{\uparrow}{y'} + \underbrace{x}_{p(x)} y = \underbrace{x}_{q(x)} \qquad y(0) = 2$

1

$\bar{x} = 0$

$\bar{y} = 2$

$\left.\begin{array}{l} p(x) = x \\ q(x) = x \end{array}\right\}$ both continuous fcts of x for all $x \in \mathbb{R}$

$\Longrightarrow$ unique soln exists for $x \in \mathbb{R}$.

In fact:

$$y(x) = 1 + e^{-\frac{1}{2}x^2}$$

is that soln.

(2) $$\underset{1}{y'} + \underbrace{\frac{1}{x}}_{p(x)} y = \underbrace{2}_{q(x)}$$

$$\left.\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 2 \end{aligned}\right\}$$ continuous fcts of x in two intervals

$I_1 = (-\infty, 0)$

$I_2 = (0, \infty)$

$\Rightarrow$ unique soln exists in either of these intervals depending on the sign of $\hat{x}$ in the IC: $y(\hat{x}) = \hat{y}$

$\hat{x} > 0$: soln ex. in I_2

$\hat{x} < 0$: soln ex. in I_1

In fact the general soln. is

$$y(x) = x + \frac{A}{x} \leftarrow \text{arb. constant.}$$

<u>Note:</u> In general this soln is singular at $x=0$.

But: For specific initial conditions e.g.:

$$y(x=1) = 1 \Rightarrow A = 0$$

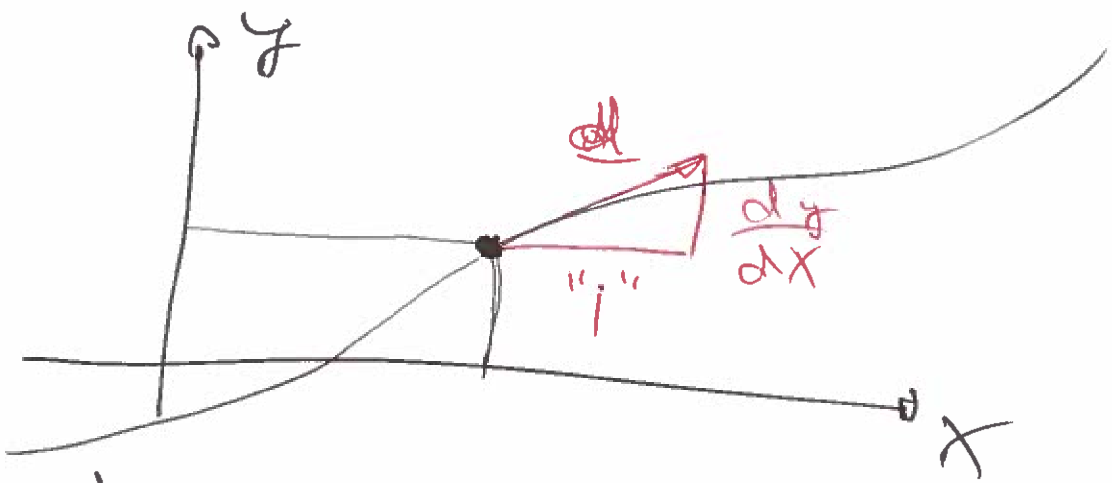
$$x = 1 > 0$$

$y(x) = x$ which exists for all $x \in \mathbb{R}$.

§2 First-order ODEs

$$y' = f(x,y) = \frac{dy}{dx}$$

(I) Graphical approach



$\frac{dy}{dx}$ is the slope of $y(x)$

$\Longrightarrow$ $f(x,y)$ defines the slope of the soln.

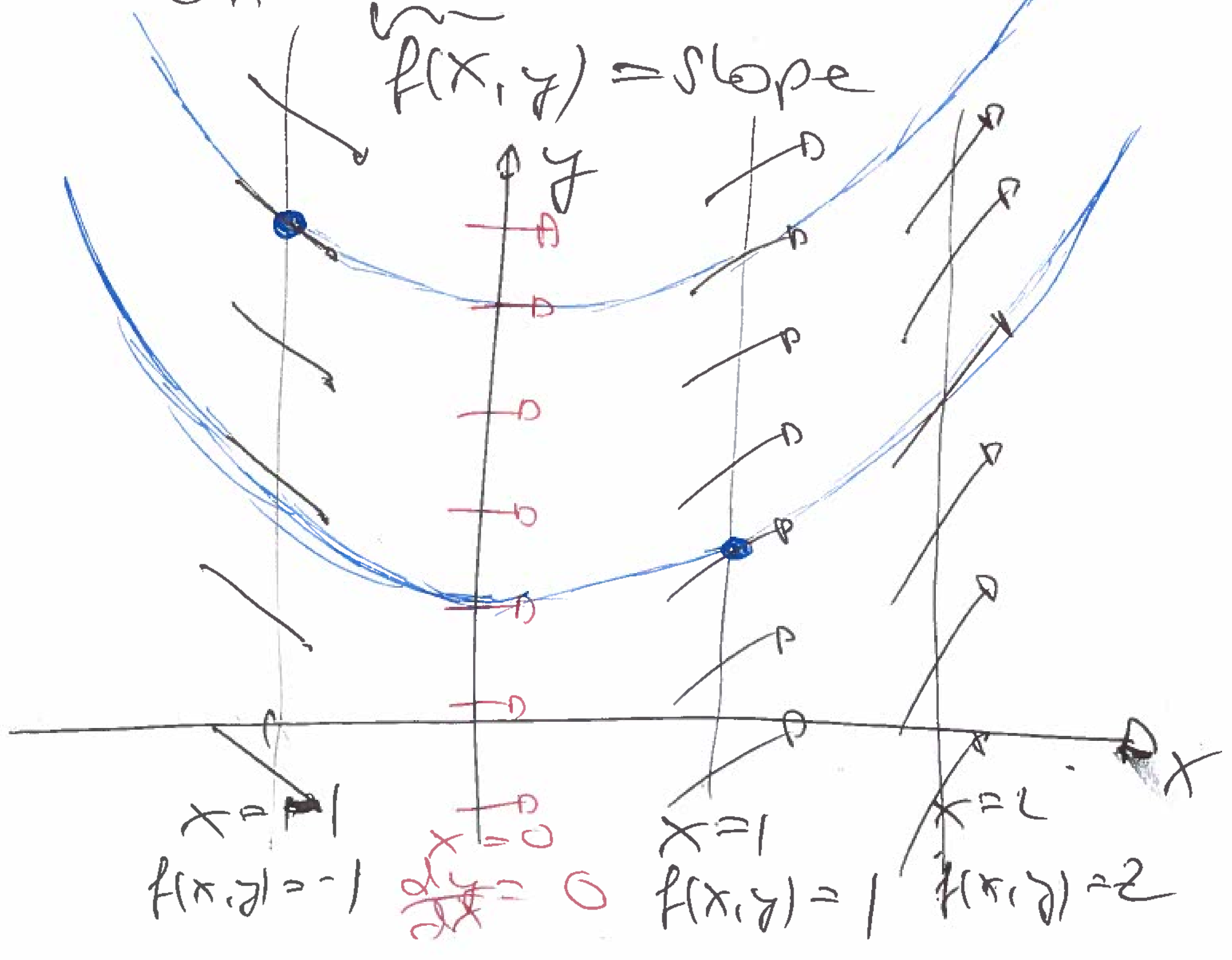
<u>Def.:</u> The "direction field" of the ODE $\frac{dy}{dx} = f(x,y)$ is the set of all vectors that have the same slope as

$$\underline{d} = \begin{pmatrix} 1 \\ \frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} 1 \\ f(x,y) \end{pmatrix}$$

Def. "Integral curves" are curves that are everywhere tangent to the direction field. Each integral curve represents a soln of the ODE.

Examples:

$$\frac{dy}{dx} = \underbrace{x}_{f(x,y)=\text{slope}}$$

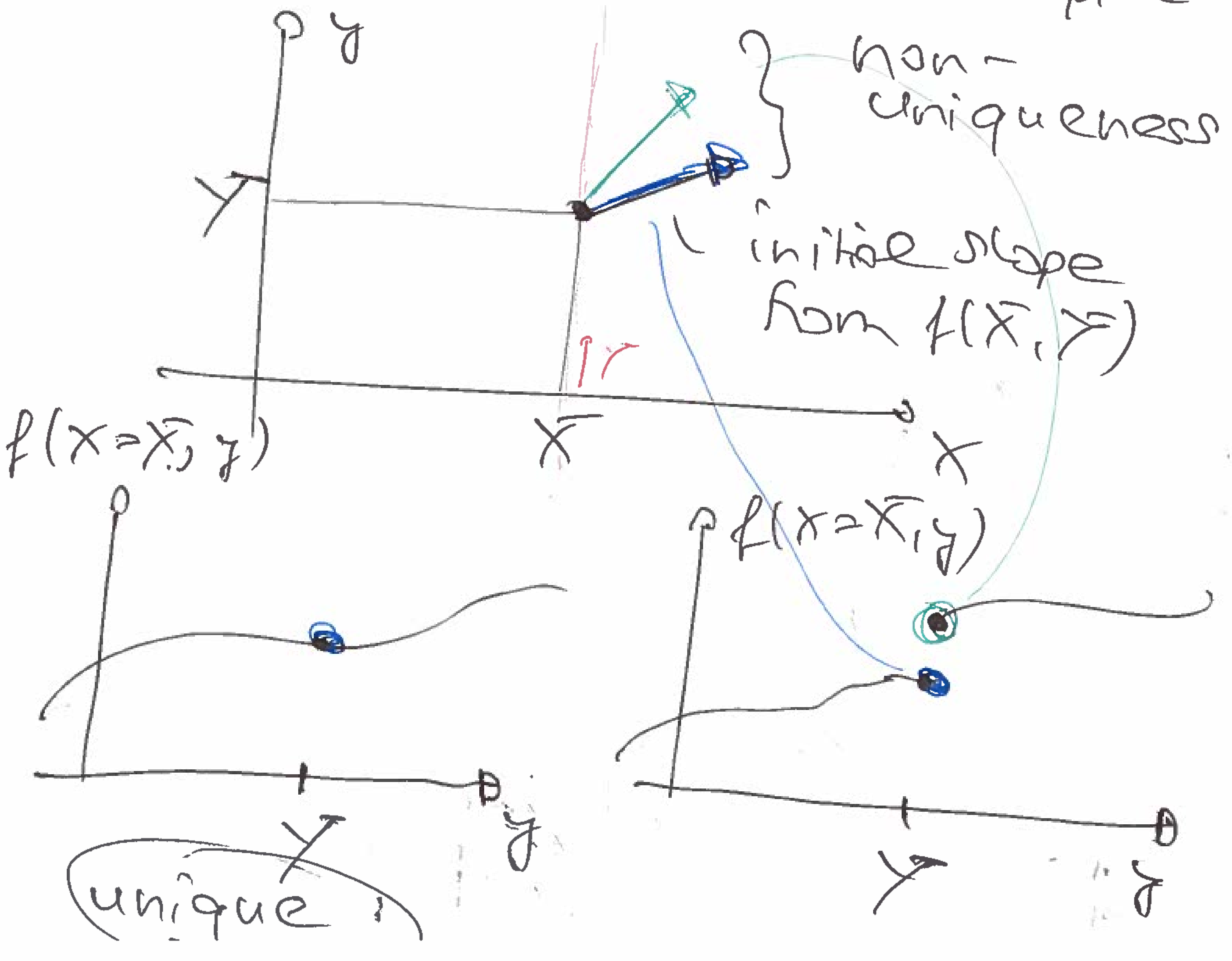


In fact, the general soln is

$$y(x) = \frac{1}{2}x^2 + C$$

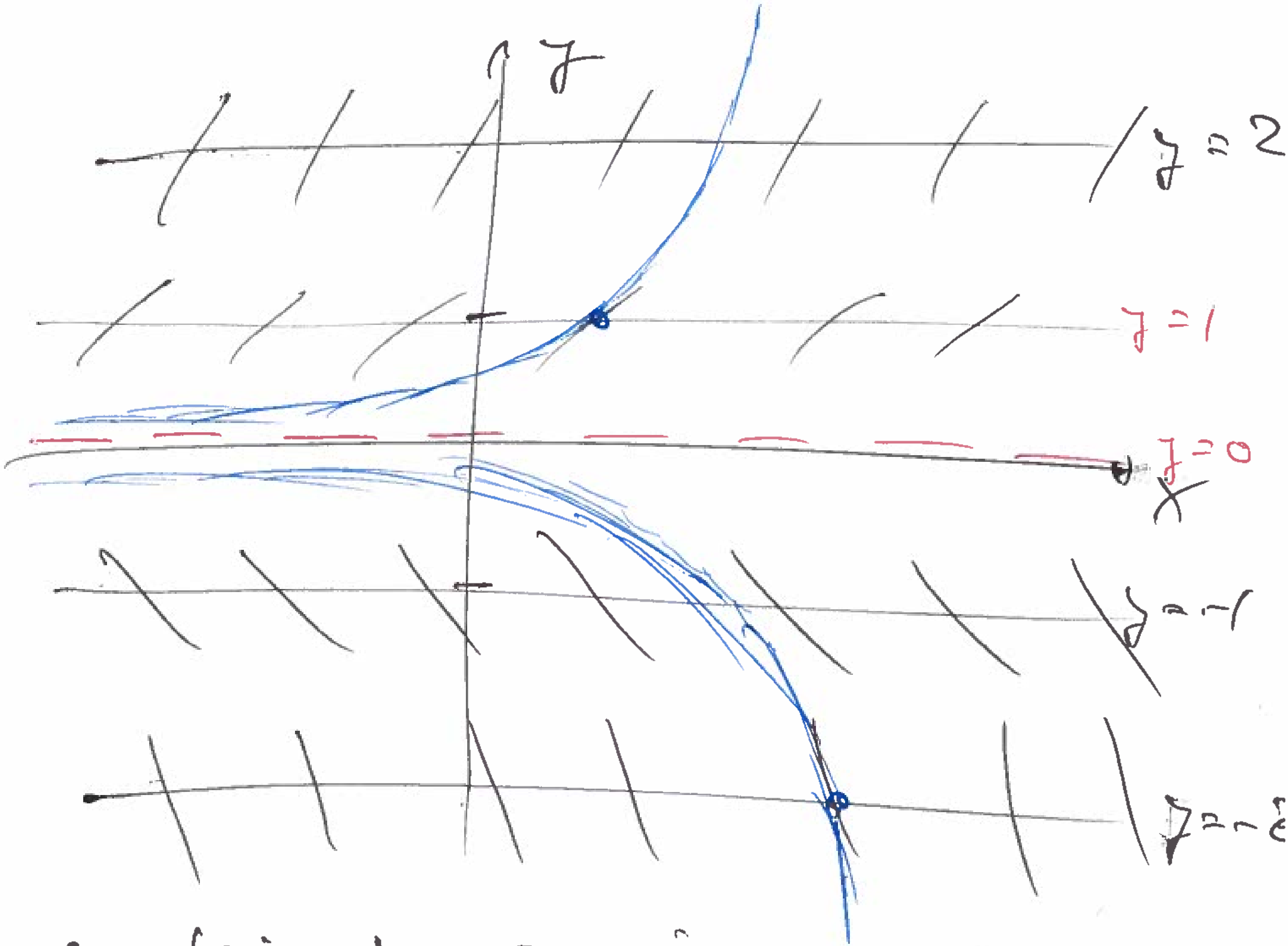
Graphical soln shows the character of the solns & illustrates E & U.

In fact, how can a solution starting from a given IC not be unique?



Example:

$$\frac{dy}{dx} = \underbrace{y}_{f(x,y) = \text{slope of } y(x)}$$



Note: $y = 0$ is a soln

In fact: Gen. soln is

$$y = A e^x$$