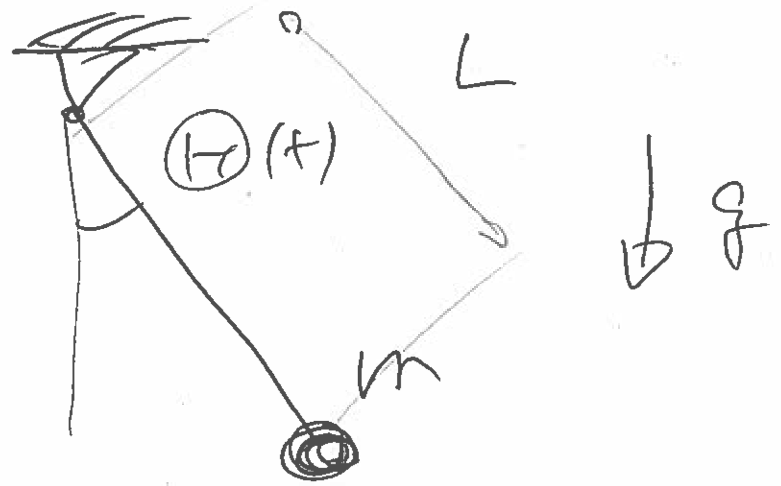


$$\boxed{\begin{aligned}&\ddot{\Theta} + \omega^2 \sin\Theta = 0\\ &\underline{\text{IC:}}\quad \Theta(t=0) = \varepsilon\\ &\qquad\ \ \dot{\Theta}(t=0) = 0\end{aligned}}$$

$$\omega^2 = \frac{g}{L}$$

<u>E&U:</u> $f(t, \Theta, \dot{\Theta}) = -\omega^2 \sin\Theta \quad (1)$

$$\ddot{\Theta} = f(t, \Theta, \dot{\Theta})$$

$$\frac{\partial f}{\partial \Theta} = -\omega^2 \cos\Theta \quad (2)$$

$$\frac{\partial f}{\partial \dot{\Theta}} = 0 \quad (3)$$

<u>Theorem:</u> (1)–(3) are continuous fcts of t, Θ & $\dot{\Theta}$

$\Rightarrow$ unique soln. exists for small t.

Existence and Uniqueness for *non-linear* 2nd order ODEs

Consider the *non-linear* second-order ODE

$$y'' = f(x, y, y') \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the function $f(x, y, y')$, are given.

Theorem

If $f(x, y, y')$ and $\frac{\partial f(x,y,y')}{\partial y}$ and $\frac{\partial f(x,y,y')}{\partial y'}$ are continuous functions of x, y and y' in a region $|x - X| < a$, $|y - Y| < b$ and $|y' - Z| < c$, where $a, b, c > 0$ then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $|x - X| < h \leq a$, where $h > 0$.

Notes:

- The statement is easily generalised to (even) higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

[Numerical] experiment: Finite-amplitude oscillation of an undamped pendulum

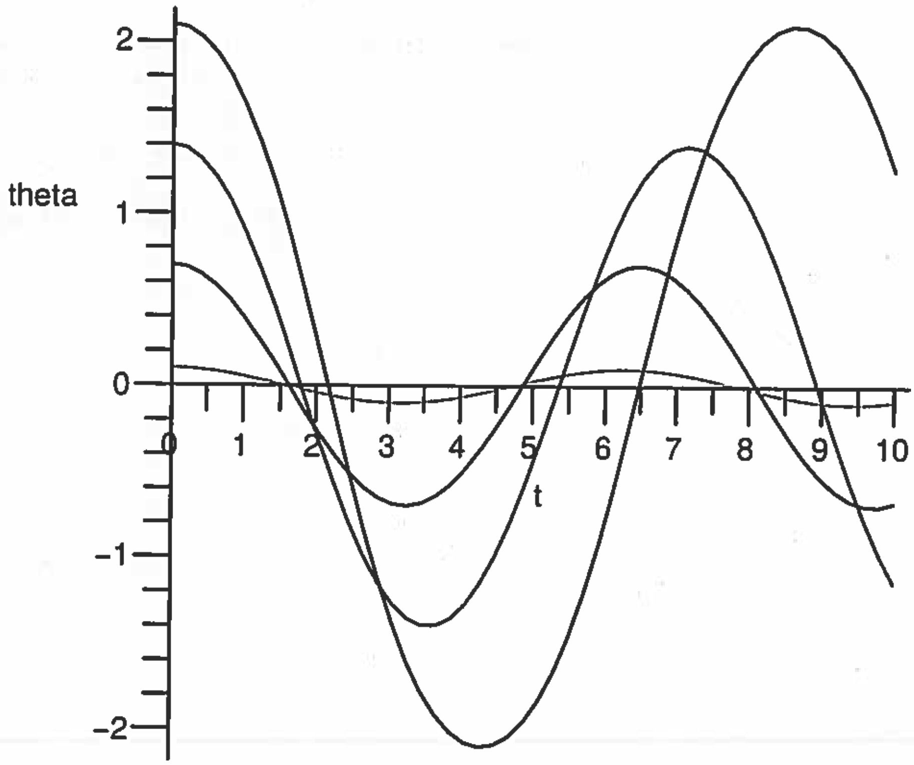
- Governing (non-linear!) ODE:

$$\ddot{\theta} + \sin\theta = 0$$

  subject to the initial conditions

$$\theta(t=0) = \epsilon \quad \text{and} \quad \dot{\theta}(t=0) = 0.$$

- Plot for $\epsilon = 0.1, 0.7, 1.4, 2.1$:



- **Observation:** Period of the oscillation increases for larger amplitudes.

Now for simplicity set $\omega^2 = 1$.

$$\boxed{\begin{aligned} &\ddot{\Theta} + \sin\Theta = 0 \\ &\Theta(t=0) = \varepsilon \\ &\dot{\Theta}(t=0) = 0 \end{aligned}}$$

we know from ~~exp~~ num./physical experiment that

$$|\Theta(t)| \leq \varepsilon$$

Now consider $|\varepsilon| \ll 1$:

$$\sin\Theta = \Theta - \frac{1}{3!}\Theta^3 + \dots$$

So in this limit the problem becomes (approximately):

$$\boxed{\begin{aligned} &\ddot{\Theta} + \Theta = 0 \\ &\Theta(t=0) = \varepsilon \\ &\dot{\Theta}(t=0) = 0 \end{aligned}} \Rightarrow \Theta(t) = \varepsilon\cos(t)$$

NOT THERE

This suggests to use an expansion of the form:

$$\Theta(t) = \cancel{\Theta_0(t)} + \varepsilon\Theta_1(t) + \varepsilon^2\Theta_2(t) + \dots \quad (*)$$

<u>Plan:</u> into IVP, expand, collect powers of ε $\Longrightarrow$ sequence of IVPs.

1^{st} step: expand $\sin\Theta$ in Taylor series:

$$\ddot{\Theta} + \underbrace{\Theta - \frac{1}{3!}\Theta^3 + \frac{1}{5!}\Theta^5 - \dots}_{\sin\Theta} = 0$$

insert $(*)$

$$\varepsilon\ddot{\Theta}_1 + \varepsilon^2\ddot{\Theta}_2 + \varepsilon^3\ddot{\Theta}_3 + \dots +$$

$$\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \dots$$

$$-\frac{1}{3!}\left(\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \dots\right)^3 + \dots = 0$$

expand $(\ldots)^3$:

$$(\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \ldots)$$
$$(\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \ldots)$$
$$(\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \ldots) =$$

$$\varepsilon^3 \Theta_1^3 + 3 \varepsilon^4 \Theta_1^2 \Theta_2 + \ldots$$

collect:

$$\boxed{\varepsilon(\ddot{\Theta}_1 + \Theta_1) + \varepsilon^2(\ddot{\Theta}_2 + \Theta_2) + \varepsilon^3\left(\ddot{\Theta}_3 + \Theta_3 - \frac{1}{6}\Theta_1^3\right) + \ldots = 0}$$

IC:

$$\Theta(t=0) = \varepsilon \Theta_1(t=0) + \varepsilon^2 \Theta_2(t=0) + \varepsilon^3 \Theta_3(t=0) + \ldots = \varepsilon$$

$$\boxed{\varepsilon(\Theta_1(t=0) - 1) + \varepsilon^2(\Theta_2(t=0)) + \varepsilon^3(\Theta_3(t=0)) + \ldots = 0}$$

$$\dot{\Theta}(t=0) = \varepsilon \dot{\Theta}_1(t=0) + \varepsilon^2 \dot{\Theta}_2(t=0) + \varepsilon^3 \dot{\Theta}_3(t=0) + \dots = 0$$

Collect powers of ε in ODE & ICs:

ε^0:

$$\ddot{\Theta}_0 + \Theta_0 = 0$$
$$\Theta_0(t=0) = 0 \Rightarrow \Theta_0(t) = 0$$
$$\dot{\Theta}_0(t=0) = 0$$

NOT SHOWN

ε^1:

$$\ddot{\Theta}_1 + \Theta_1 = 0$$
$$\Theta_1(t=0) = 1 \qquad \Theta_1(t) = \cos(t)$$
$$\dot{\Theta}_1(t=0) = 0$$

ε^2:

$$\ddot{\Theta}_2 + \Theta_2 = 0$$
$$\Theta_2(t=0) = 0 \Rightarrow \Theta_2(t) = 0$$
$$\dot{\Theta}_2(t=0) = 0$$

$$\boxed{\begin{aligned} \varepsilon^3:\quad & \ddot{\Theta}_3 + \Theta_3 = \frac{1}{6}\Theta_1^3 \\ & \Theta_3(t=0) = 0 \\ & \dot{\Theta}_3(t=0) = 0 \end{aligned}}$$

$\Theta_1 = \cos(t)$

$\vdots$

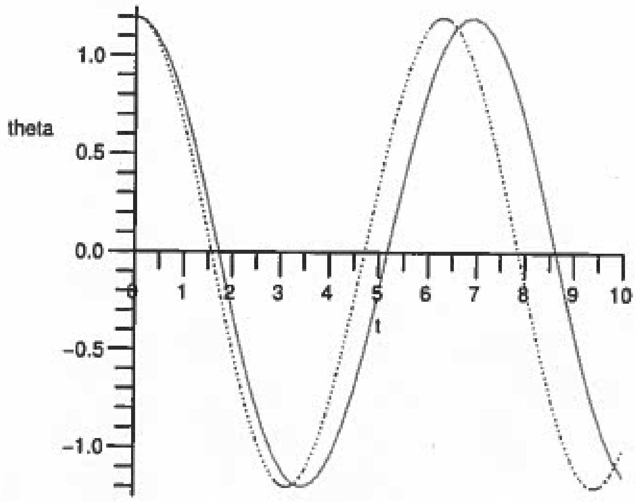
$$\Theta_3(t) = \frac{1}{192}\left(\cos(t) - \cos(3t)\right) + \frac{1}{16} t \sin(t)$$

etc.

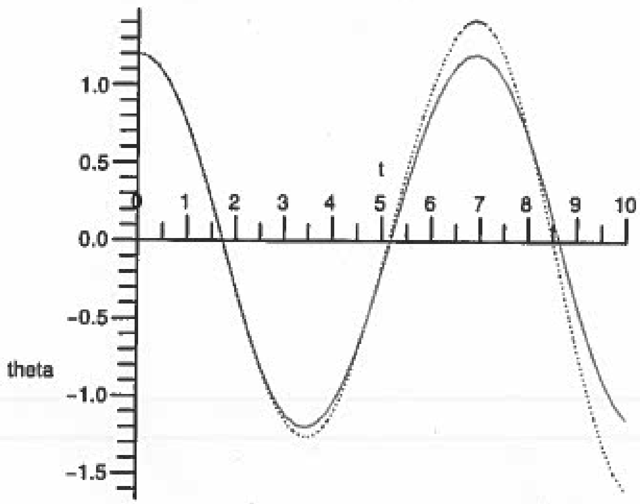
Key feature: we solve a sequence of linear problems!

Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$

- One-term perturbation solution (red), exact solution (green):

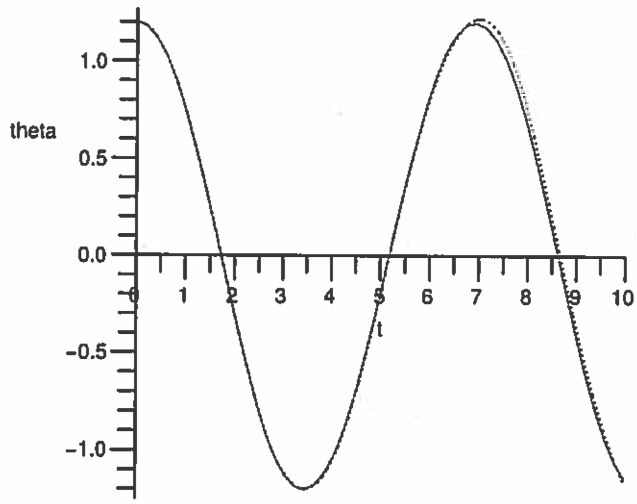


- Two-term perturbation solution (red), exact solution (green):

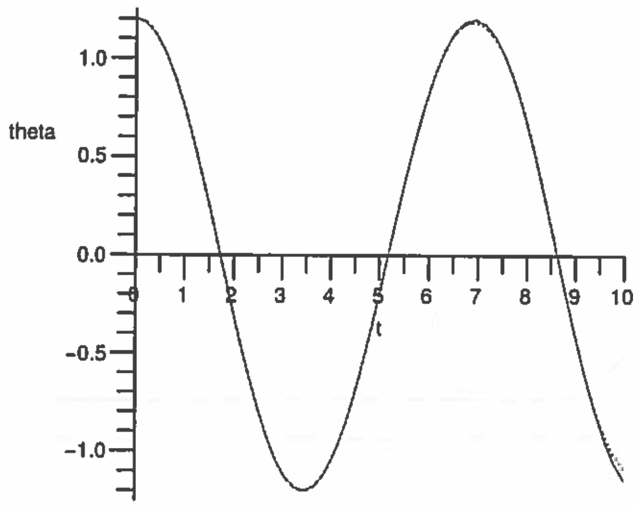


Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$ (cont.)

- Three-term perturbation solution (red), exact solution (green):

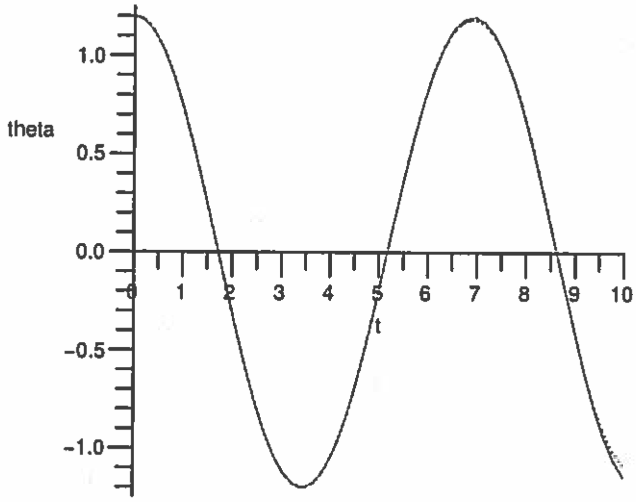


- Four-term perturbation solution (red), exact solution (green):

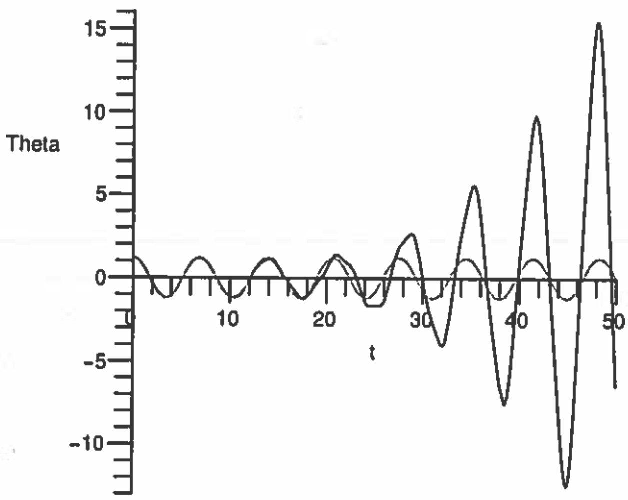


Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$ (cont.)

- Four-term perturbation solution (red), exact solution (green):



- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:



“Multinomial expansions”

- One tedious task that one tends to face regularly when using perturbation methods is that of raising a power series in ϵ to some integer power

$$S = \left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^n, \tag{1}$$

  and collecting the terms multiplied by the same power of ϵ, i.e. re-writing S in the form

$$S = S_0(x_0) + \epsilon\, S_1(x_0, x_1) + \epsilon^2\, S_2(x_0, x_1, x_2) + \ldots \tag{2}$$

  where the functions $S_i(x_0, x_1, \ldots)$ do not depend on ϵ.

- Formally, the expansion of S may be obtained by using the “multinomial series” (a generalisation of the binomial series) as

$$\left(a_1 + a_2 + \ldots + a_k\right)^n = \sum_{\substack{n_1, n_2, n_3, \ldots, n_k \in \mathbb{N}_0 \\ n_1 + n_2 + \ldots + n_k = n}} \frac{n!}{n_1!\, n_2! \ldots n_k!}\, a_1^{n_1}\, a_2^{n_2} \ldots a_k^{n_k}$$

  see, e.g. `http://mathworld.wolfram.com/MultinomialSeries.html`

- However, we usually only need the first few terms in (2) for low-ish powers of n. Here they are:

$$\begin{aligned}
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^2 &= \left(x_0^2\right) + \epsilon\left(2x_0\, x_1\right) + \epsilon^2\left(x_1^2 + 2x_0\, x_2\right) + \ldots \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^3 &= \left(x_0^3\right) + \epsilon\left(3x_0^2\, x_1\right) + \epsilon^2\left(3x_0 x_1^2 + 3x_0^2\, x_2\right) + \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^4 &= \left(x_0^4\right) + \epsilon\left(4x_0^3\, x_1\right) + \epsilon^2\left(4x_0^3 x_2 + 6x_0^2\, x_1^2\right) +
\end{aligned}$$

- **Exercise:** Convince yourself that you understand how these terms arise. **Hint:** Either use the multinomial series given above, or write S explicitly as a product of n power series [e.g. for $n = 2:\ \ S = (x_0 + \epsilon\, x_1 + \ldots)(x_0 + \epsilon\, x_1 + \ldots)$] and inspect which combination of terms gives rise to what powers of ϵ.

- **Relax!** In an exam these expressions would be provided!