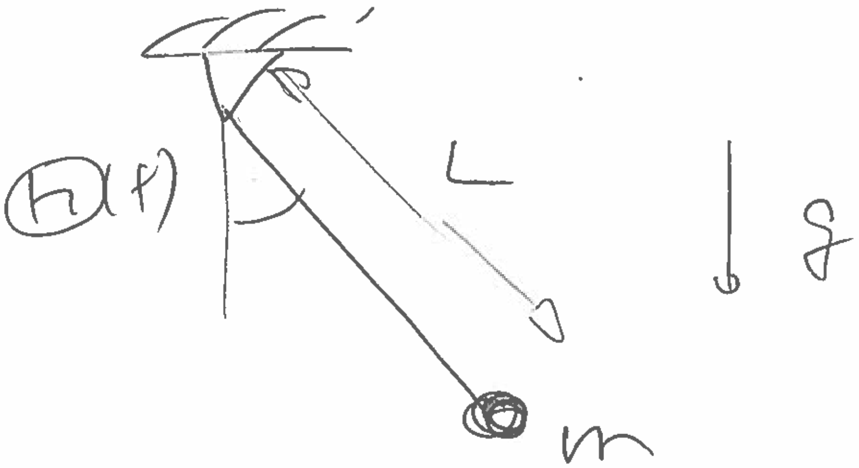


$$\omega^2 = \frac{g}{L}$$

$$\boxed{\begin{aligned} &\ddot{\Theta} + \omega^2 \sin\Theta = 0 \\ &\Theta(t=0) = \varepsilon \\ &\dot{\Theta}(t=0) = 0 \end{aligned}}$$

IVP

$$1 \cdot \ddot{\Theta} + \omega^2 \frac{\sin\Theta}{\Theta}\Theta = 0$$

E&U:

$$\ddot{\Theta} = \underbrace{-\omega^2 \sin\Theta}_{f(t,\Theta,\dot{\Theta}) = -\omega^2 \sin\Theta}$$

f & $\frac{\partial f}{\partial \Theta}$ & $\frac{\partial f}{\partial \dot{\Theta}} = 0$ are continuous fcts

$\Longrightarrow$ E&U near $t=0$.

Existence and Uniqueness for *non-linear* 2nd order ODEs

Consider the *non-linear* second-order ODE

$$y'' = f(x, y, y') \tag{1}$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \tag{2}$$

where the constants X, Y and Z, and the function $f(x, y, y')$, are given.

Theorem

If $f(x, y, y')$ and $\frac{\partial f(x,y,y')}{\partial y}$ and $\frac{\partial f(x,y,y')}{\partial y'}$ are continuous functions of x, y and y' in a region $0 < |x - X| < a$, $0 < |y - Y| < b$ and $0 < |y' - Z| < c$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

Notes:

- The statement is easily generalised to (even) higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

Let's try perturbation methods for the case ~~...~~ ε small. (2

<u>Observation:</u>

Starting from $\Theta(t=0) = \varepsilon$:

$$|\Theta(t)| \leq \varepsilon$$

This suggests Taylor expanding

$$\sin\Theta = \Theta - \frac{1}{3!}\Theta^3 + \frac{1}{5!}\Theta^5 - \dots$$

Naive: $|\Theta| \ll \varepsilon \ll 1$ only use 1st term: $\sin\Theta \approx \Theta$

$$\ddot{\Theta} + \omega^2\Theta = 0$$

in this case $\Theta(t) = \varepsilon\cos(\omega t)$

$\Longrightarrow$ As assumed max. size of $\Theta(t)$ is ε.

This suggests the ansatz:

$$\boxed{\Theta(t) = \varepsilon\Theta_1(t) + \varepsilon^2\Theta_2(t) + \ldots}$$

omit $\varepsilon^0\Theta_0(t)$

[EXERCISE: Retain $\Theta_0(t)$ in ansatz & repeat analysis to find $\Theta_0(t) = 0$]

Into ODE:

$$\ddot{\Theta} + \omega^2\underbrace{\left(\Theta - \frac{1}{3!}\Theta^3 + \frac{1}{5!}\Theta^5 - \ldots\right)}_{\sin\Theta} = 0$$

(Set $\omega^2 = 1$ to save writing)

$$\underbrace{\varepsilon\ddot{\Theta}_1 + \varepsilon^2\ddot{\Theta}_2 + \varepsilon^3\ddot{\Theta}_3 + \ldots}_{\ddot{\Theta}} +$$

$$\underbrace{\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \ldots}_{\Theta} +$$

$$-\frac{1}{6}\underbrace{\left(\varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \ldots\right)^3}_{\Theta^3} + \ldots = 0$$

To expand Θ^3: (4)

$$\left(\varepsilon\Theta_1+\varepsilon^2\Theta_2+\varepsilon^3\Theta_3+\dots\right)\left(\varepsilon\Theta_1+\varepsilon^2\Theta_2+\varepsilon^3\Theta_3+\dots\right)\left(\varepsilon\Theta_1+\varepsilon^2\Theta_2+\varepsilon^3\Theta_3+\dots\right)=$$

$$=\varepsilon^3\Theta_1^3+\varepsilon^4 3\Theta_1^2\Theta_2+\dots$$

into ODE...

$$\ddot{\varepsilon\Theta_1}+\varepsilon^2\ddot{\Theta}_2+\varepsilon^3\ddot{\Theta}_3+\dots+$$
$$\varepsilon\Theta_1+\varepsilon^2\Theta_2+\varepsilon^3\Theta_3+\dots+$$
$$-\frac{1}{6}\left(\varepsilon^3\Theta_1^3+3\varepsilon^4\Theta_1^2\Theta_2+\dots\right)+\dots=0$$

Collect:

$$\varepsilon\left(\ddot{\Theta}_1+\Theta_1\right)+$$
$$\varepsilon^2\left(\ddot{\Theta}_2+\Theta_2\right)+$$
$$\varepsilon^3\left(\ddot{\Theta}_3+\Theta_3-\frac{1}{6}\Theta_1^3\right)+\dots=0$$

IC:

$$\boxed{\Theta(0) = \varepsilon = \varepsilon\Theta_1(0) + \varepsilon^2\Theta_2(0) + \dots}$$

$$\boxed{\varepsilon\left(\Theta_1(0) - 1\right) + \varepsilon^2\Theta_2(0) + \varepsilon^3\Theta_3(0) + \dots = 0}$$

$$\dot{\Theta}(0) = \boxed{0 = \varepsilon\dot{\Theta}_1(0) + \varepsilon^2\dot{\Theta}_2(0) + \varepsilon^3\dot{\Theta}_3(0) + \dots = 0}$$

ε^1:

$$\ddot{\Theta}_1 + \Theta_1 = 0$$
$$\Theta_1(0) = 1$$
$$\dot{\Theta}_1(0) = 0$$
$$\Rightarrow \boxed{\Theta_1(t) = \cos(t)}$$

ε^2:

$$\ddot{\Theta}_2 + \Theta_2 = 0$$
$$\Theta_2(0) = 0$$
$$\dot{\Theta}_2(0) = 0$$
$$\Rightarrow \boxed{\Theta_2(t) = 0}$$

ε^3:

$$\ddot{\Theta}_3 + \Theta_3 = \frac{1}{6}\Theta_1^3$$
$$\Theta_2(0) = 0$$
$$\dot{\Theta}_3(0) = 0$$

$$\ddot{\Theta}_3 + \Theta_3 = \frac{1}{6}\cos^3(t)$$

$$\Theta_{3H} = A\cos t + B\sin t$$

~~4~~ <u>SKETCH:</u>

<u>Θ_{3p}:</u> Naive

$$\Theta_{3p}(t) = C\cos^3(t)$$

will not work because:

$$\cos^3(t) = \frac{1}{4}\left(\cos(3t) + 3\cos(t)\right)$$

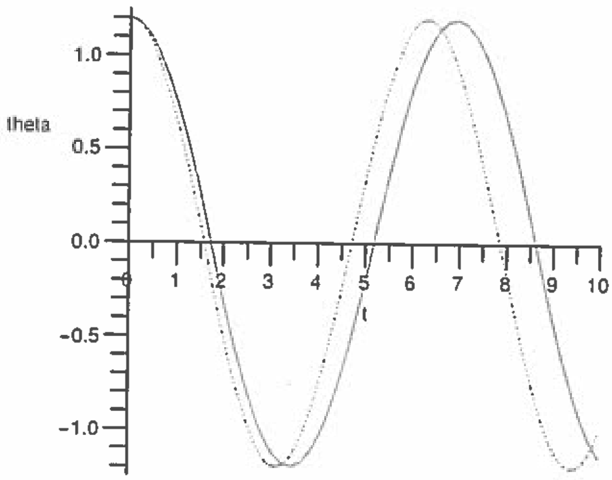
solves homog ODE

⋮ EXERCISE

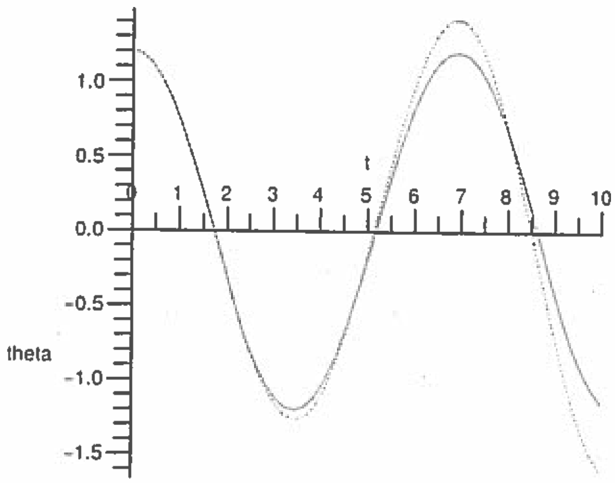
$$\Theta_3(t) = \frac{1}{192}\left(\cos(t) - \cos(3t)\right) + \frac{1}{16}t\sin t$$

Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$

- One-term perturbation solution (red), exact solution (green):

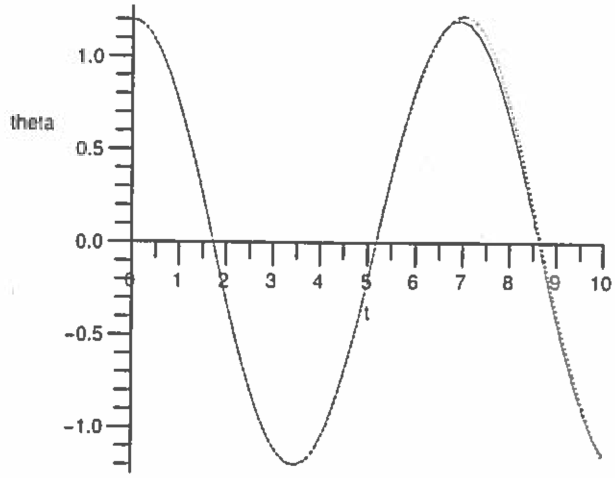


- Two-term perturbation solution (red), exact solution (green):

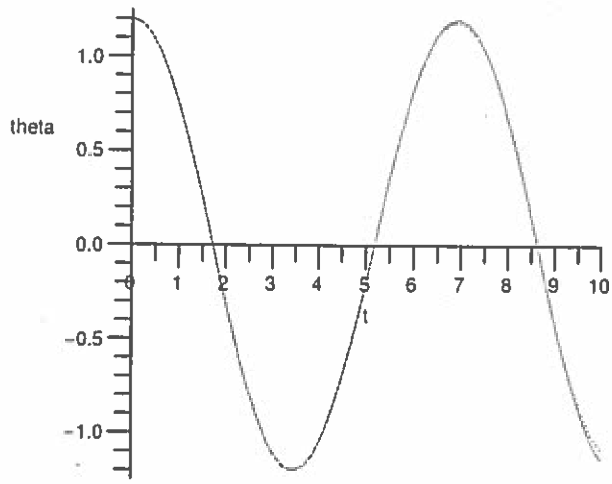


Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$ (cont.)

- Three-term perturbation solution (red), exact solution (green):

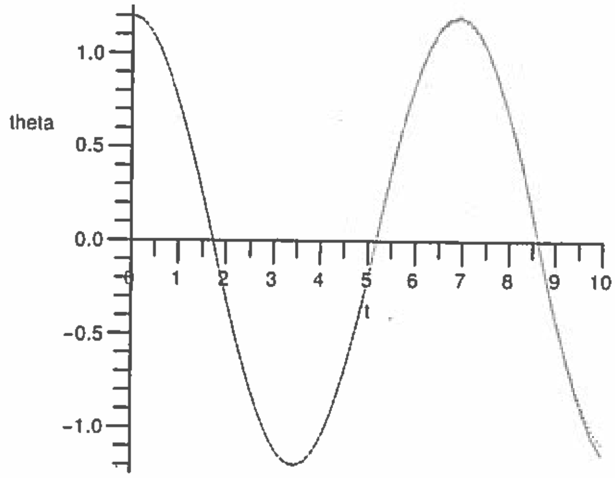


- Four-term perturbation solution (red), exact solution (green):

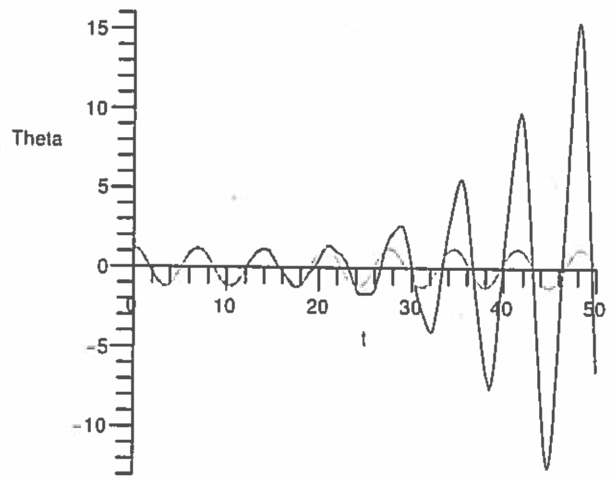


Comparison between perturbation solution and "exact" solution for $\epsilon = 1.2$ (cont.)

- Four-term perturbation solution (red), exact solution (green):



- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:



"Multinomial expansions"

- One tedious task that one tends to face regularly when using perturbation methods is that of raising a power series in ϵ to some integer power

$$S = \left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^n, \tag{1}$$

  and collecting the terms multiplied by the same power of ϵ, i.e. re-writing S in the form

$$S = S_0(x_0) + \epsilon\, S_1(x_0, x_1) + \epsilon^2\, S_2(x_0, x_1, x_2) + \ldots \tag{2}$$

  where the functions $S_i(x_0, x_1, \ldots)$ do not depend on ϵ.

- Formally, the expansion of S may be obtained by using the "multinomial series" (a generalisation of the binomial series) as

$$\left(a_1+a_2+\ldots+a_k\right)^n = \sum_{\substack{n_1, n_2, n_3, \ldots, n_k \in \mathbb{N}_0 \\ n_1+n_2+\ldots+n_k = n}} \frac{n!}{n_1!\, n_2! \ldots n_k!}\, a_1^{n_1}\, a_2^{n_2} \ldots a_k^{n_k}$$

  see, e.g. `http://mathworld.wolfram.com/MultinomialSeries.html`

- However, we usually only need the first few terms in (2) for low-ish powers of n. Here they are:

$$\begin{aligned}
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^2 &= \left(x_0^2\right) + \epsilon\left(2x_0\, x_1\right) + \epsilon^2\left(x_1^2 + 2x_0\, x_2\right) + \ldots \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^3 &= \left(x_0^3\right) + \epsilon\left(3x_0^2\, x_1\right) + \epsilon^2\left(3x_0 x_1^2 + 3x_0^2\, x_2\right) + \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots\right)^4 &= \left(x_0^4\right) + \epsilon\left(4x_0^3\, x_1\right) + \epsilon^2\left(4x_0^3 x_2 + 6x_0^2\, x_1^2\right) +
\end{aligned}$$

- **Exercise:** Convince yourself that you understand how these terms arise. **Hint:** Either use the multinomial series given above, or write S explicitly as a product of n power series [e.g. for $n = 2:\ \ S = (x_0 + \epsilon\, x_1 + \ldots)(x_0 + \epsilon\, x_1 + \ldots)$] and inspect which combination of terms gives rise to what powers of ϵ.

- **Relax!** In an exam these expressions would be provided!