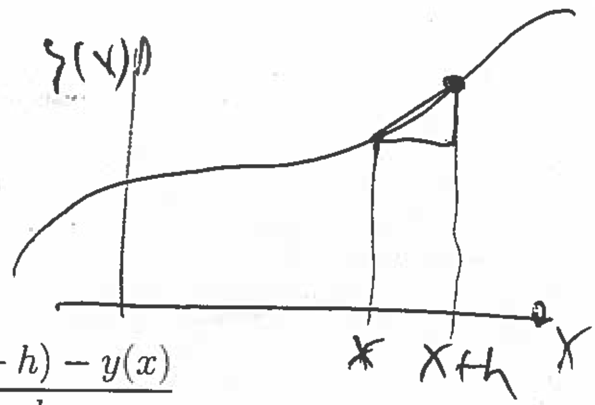
INTRODUCTION

Notation, Definitions and "What are the issues?"

The Derivative

Given a function

$$y(x)$$



where

- x is the independent variable,
- y is the dependent variable,

the derivative is defined as

$$y'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$
$$= \lim_{h \to 0} \frac{y(x) - y(x-h)}{h}.$$

The derivative is not defined at points where the "right" and "left" limits do not converge to the same value. For instance, $y(x) = |x|$ does not have a derivative at $x = 0$.

Higher Derivatives

Higher derivatives are defined recursively

$$y''(x) = \frac{d^2y(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy(x)}{dx}\right)$$
$$y'''(x) = \frac{d^3y(x)}{dx^3} = \frac{d}{dx}\left(\frac{d^2y(x)}{dx^2}\right)$$
etc.

...provided the lower-order derivatives are sufficiently smooth for the higher derivatives to exist.

Notation

- Dash notation:

$$\frac{d}{dx}(.) = (.)'$$
$$\frac{d^2}{dx^2}(.) = (.)''$$
$$\frac{d^n}{dx^n}(.) = (.)^{(n)}$$

- Dot notation: For time-dependent problems, where t is the independent variable, dots are often used to indicate derivatives.

$$x(t)$$
$$\frac{dx}{dt} = \dot{x}(t)$$
$$\frac{d^2x}{dt^2} = \ddot{x}(t)$$

- The dependence on the independent variable may be suppressed. For instance, instead of

$$y'(x) + p(x)\,y(x) = r(x)$$

we can simply write

$$y' + p(x)\,y = r(x)$$

because it's "obvious" that y is a function of x.

Ordinary differential equations

Definition:

- An n-th order ordinary differential equation (ODE) for $y(x)$ has the general form

$$\mathcal{F}\big(x, y(x), y'(x), ..., y^{(n)}(x)\big) = 0, \tag{1}$$

  i.e. it relates the (unknown) function, $y(x)$ to x and its 1st, 2nd, ..., nth derivatives.

- Often the implicit form given above can be solved for $y(x)$, allowing the ODE to be written in explicit form as:

$$y^{(n)}(x) = f\big(x, y(x), y'(x), ..., y^{(n-1)}(x)\big). \tag{2}$$

Solutions:

- A solution of the ODE (1) [or (2)] in an interval

$$I = \{x \mid a < x < b\}$$

  is *any* function $\phi(x)$ for which

$$\mathcal{F}\big(x, \phi(x), \phi'(x), ..., \phi^{(n)}(x)\big) = 0 \quad \forall x \in I. \tag{3}$$

Notes:

- The statement already suggests that there may be multiple solutions.
- Furthermore, solutions might not exist for all values of x.
- In fact, there might not be a solution at all!

Two properties worth looking out for...

1. Linearity

- An ODE is linear if

$$\mathcal{F}\big(x, y(x), y'(x), ..., y^{(n)}(x)\big) = 0,$$

is linear in y and all its derivatives.

- Linear ODEs can be written as

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y = b(x)$$

where $a_i(x)$ $(i = 0, ..., n)$ and $b(x)$ are given functions.

2. Autonomous ODEs

- An ODE is autonomous if it has the form

$$\mathcal{F}\big(y(x), y'(x), ..., y^{(n)}(x)\big) = 0,$$

i.e. if the independent variable, x, does not appear explicitly.

Examples

Ex: $y'' - \tan(x)\, y' = -2\sin(x)$

is 2^{nd} order, linear
non-autonomous ODE
for $y(x)$.

$y(x) = \sin(x)$ is a soln.

$y'(x) = \cos(x)$

$y''(x) = -\sin(x)$

$y'' - \tan(x)\, y' = -2\sin(x)$

$-\sin(x) - \frac{\sin(x)}{\cos(x)} \cos(x) \stackrel{?}{=} -2\sin(x)$ ✓

Are there other solns?

Ex: $\frac{1}{24} y''' + y^{1/4} = 2x$

is 3^{rd} order, non-autonomous nonlinear ODE for $y(x)$.

A soln is $y(x) = x^4$

$y''' = 4 \cdot 3 \cdot 2 x = 24x$

$y^{1/4} = x$

$\frac{1}{24} y''' + y^{1/4} = 2x$

$\frac{1}{24} 24x + x \overset{?}{=} 2x$ ✓

Are there others?

Ex: $\frac{1}{24} y''' - y^{1/4} = 0$

3rd order, nonlinear autonomous ODE for $y(x)$.

$y(x) = x^4$ solves this too

So: Easy to check if a candidate soln. is a soln.

Q: How does one ~~[illegible]~~ address this systematically?

- Existence: Are there any solns?
- Uniqueness: Is there only one soln?

Existence:

If the ODE describes a physical problem (& if it does so correctly) then we expect the soln. to exist.

However existence is not obvious:

Ex:
$$y' + \frac{1}{y'} = 0$$

$$(y')^2 = -1$$

$$y' = \sqrt{-1} = i$$

There are no real valued solns!

Uniqueness:

Ex: $$\frac{d^2y}{dx^2} = 0$$

2nd order, linear autonomous ODE for $y(x)$.

Integrate twice:

$$y(x) = Ax + B$$

for any values of these two constants.

Need to apply two constraints to obtain a unique soln.

Ex: $$\left.\begin{aligned} y(0) &= 0 \\ \left.\frac{dy}{dx}\right|_{x=0} &= 3 \end{aligned}\right\} \quad y(x) = 3x$$

Ex:
$$\left.\begin{aligned} y(0) &= 1 \\ y(1) &= 10 \end{aligned}\right\} y(x) = 9x + 1$$

Boundary and initial conditions

> An m-th order ODE must be augmented by m constraints (in the form of "boundary" or "initial" conditions) if there is to be a unique solution.

Notes:

- This is a necessary, not a sufficient condition: Even if an m-th order ODE is augmented by m constraints, there may be multiple (or no!) solutions.

Initial conditions (ICs)

- If all constraints are applied at the same value of the independent variable, we refer to them as *initial conditions*.

Boundary conditions (BCs)

- If the constraints are applied at multiple values of the independent variable (typically at the ends of the interval I in which the solution is sought), we refer to them as *boundary conditions*.

Boundary and initial value problems

Initial value problems (IVPs)

$$\text{“IVP = ODE + ICs”}$$

- Initial value problems (IVPs) typically describe evolution processes in which the initial state (at time $t = t_0$, say) of a system is characterised by the initial conditions, while the ODE describes the dynamics of its subsequent evolution.

Boundary value problems (BVPs)

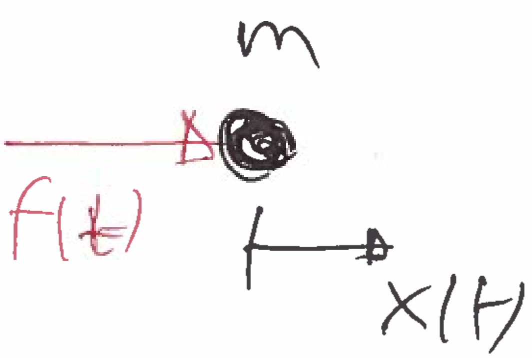
$$\text{“BVP = ODE + BCs”}$$

- Boundary value problems (BVPs) typically describe spatial problems in which the boundary conditions describe the state of the system on the domain boundaries, while the ODE governs its behaviour in the interior of the domain.

Example: IVP

1D motion of a particle of mass m, subject to a time-dependent force $f(t)$ is described by Newton's law:

$$m \frac{d^2x}{dt^2} = f(t) \qquad 2^{nd} \text{ order}$$



IC: $x(t=0) = x_0$ initial position

$\left.\frac{dx}{dt}\right|_{t=0} = v_0$ initial velocity

Exercise: $f(t) = F_0 = \text{const}$

$$X(t) = \frac{1}{2}\frac{F_0}{m}t^2 + At + B$$

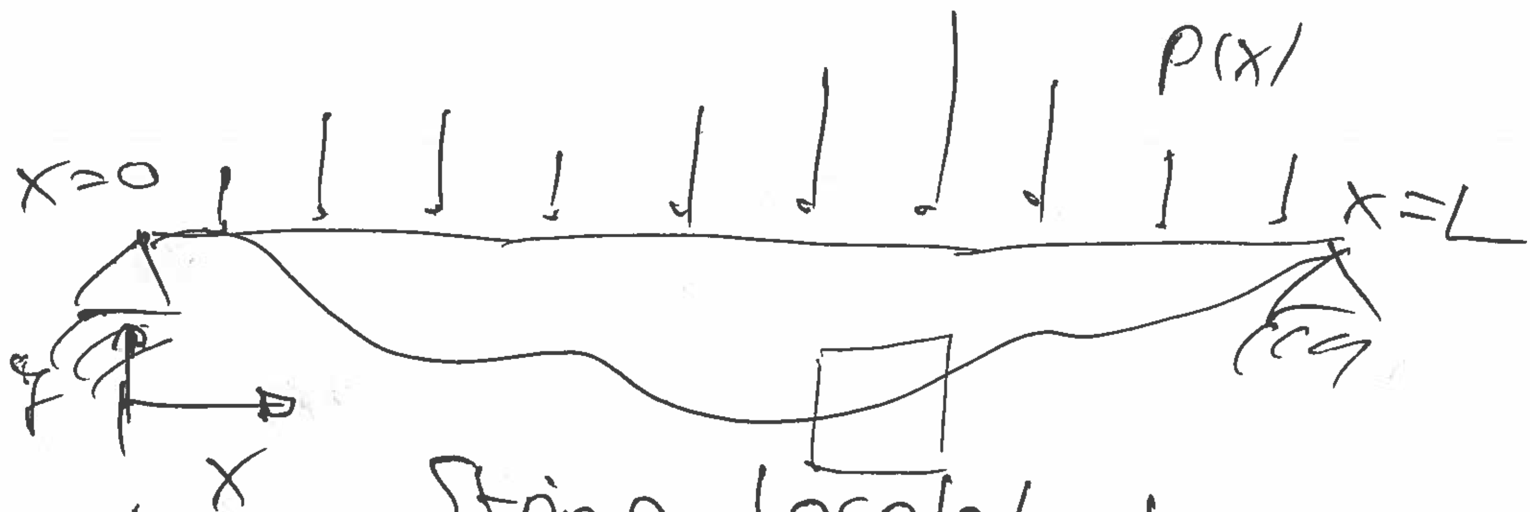
arbitrary constants

Apply ICs:

$$x(t) = \frac{1}{2}\frac{F_0}{m}t^2 + v_0 t + x_0$$

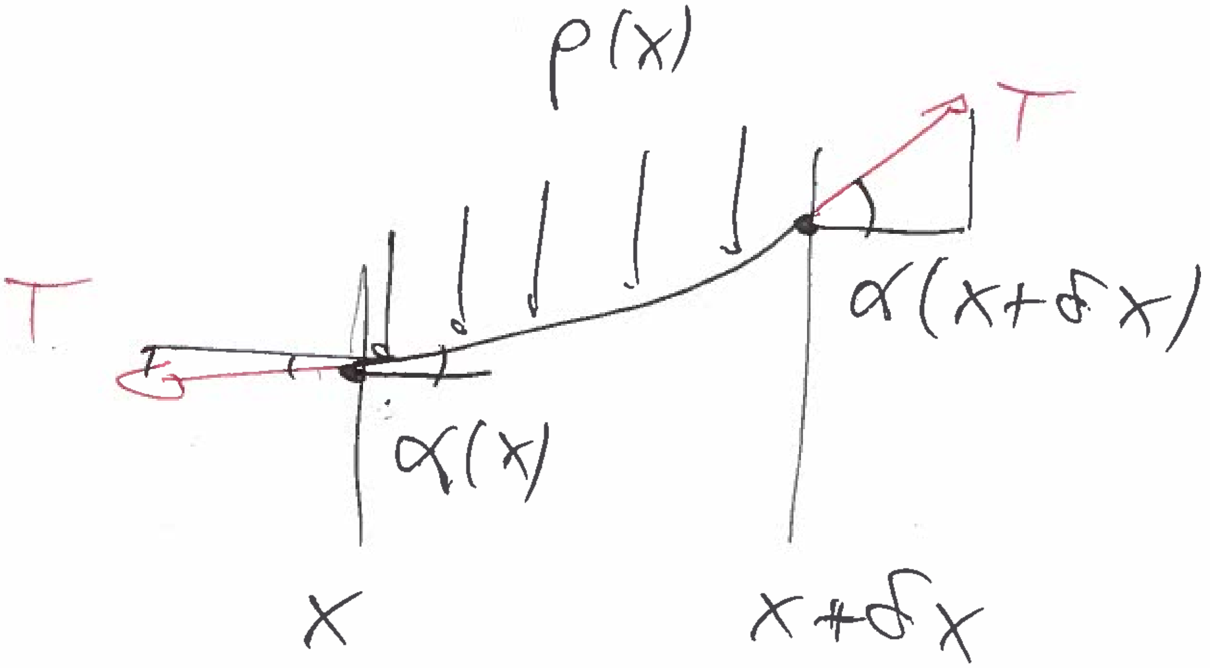
<u>Example</u> BVP:

Elastic string subject to a constant tension T, loaded by a transverse pressure $p(x)$.



String located at ~~$g(x)$~~ $y = h(x)$

Balance of forces on a small segment of string:



$$\underbrace{p(x)\,\delta x}_{\text{downward force due to pressure}} = \underbrace{T\sin(\alpha(x+\delta x)) - T\sin(\alpha(x))}_{\text{net upwards force due to tension.}}$$

Now assume: small slope $\|\alpha\| \ll 1$

$$\sin\alpha \approx \tan\alpha = \frac{dh}{dx}$$

$$p(x)\,\delta x = T\left(\left.\frac{dh}{dx}\right|_{x+\delta x} - \left.\frac{dh}{dx}\right|_{x}\right)$$

$$p(x) = T \lim_{\delta x \to 0}\left(\frac{\left.\frac{dh}{dx}\right|_{x+\delta x} - \left.\frac{dh}{dx}\right|_{x}}{\delta x}\right)$$

$$= T\frac{d}{dx}\left(\frac{dh}{dx}\right)$$

$$\boxed{p(x) = T\frac{d^2h}{dx^2}}$$

2^{nd} order ODE for $h(x)$

<u>BCs:</u> $h(x=0) = 0$

$h(x=L) = 0$

Example: $\rho(x) = \rho_0 = \text{const.}$

$$h(x) = \frac{1}{2}\frac{\rho_0}{T}x^2 + Ax + B$$

2 arbitrary constants

Apply BCs:

$$h(x) = \frac{1}{2}\frac{\rho_0}{T}\left(x^2 - Lx\right)$$