§6 Perturbation methods

- Problem has small parameter ε
- Problem is "easy" for $\varepsilon = 0$

$$x^2 + \varepsilon x - 1 = 0$$

$$x(\varepsilon) = -\frac{1}{2}\varepsilon \pm \sqrt{\frac{1}{4}\varepsilon^2 + 1}$$

small ε: Taylor expand

$$x(\varepsilon) = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \dots$$

Ansatz:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

into eqn & collect powers of ε

$$(x_0^2 - 1) + \varepsilon(2x_0 x_1 + x_0) + \varepsilon^2(x_1^2 + 2x_0 x_2 + x_1) + \dots = 0$$

$\vdots$

$$x(\varepsilon) = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \dots$$

Note the structure of the eqns:

- Lowest-order eqn is the full eqn for $\varepsilon = 0$, which we assumed was "easy".
- Higher-order eqns provide corrections via a hierarchy of problems.
- Eqn. itself is satisfied ~~[illegible]~~ up to a quantifiable error.

Questions:

- How do we "guess" the right form of the expansion?
- Will it always work?

An ODE example

Damped oscillator with weak damping.

$$\boxed{\begin{aligned} &\ddot{x} + \varepsilon \dot{x} + x = 0 \\ &x(t=0) = 1 \\ &\dot{x}(t=0) = 0 \end{aligned}} \Longrightarrow x(t;\varepsilon)$$

Assume we don't know the soln. Note: for $\varepsilon = 0$: $x(t, \varepsilon=0) = \cos(t)$

Ansatz for small ε:

$$x(t;\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$$

into problem:

$$\dot{x}(t) = \dot{x}_0(t) + \varepsilon \dot{x}_1(t) + \varepsilon^2 \dot{x}_2(t) + \dots$$

$$\ddot{x}(t) = \ddot{x}_0(t) + \varepsilon \ddot{x}_1(t) + \varepsilon^2 \ddot{x}_2(t) + \dots$$

into ODE

$$\underbrace{\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots}_{\ddot{x}} +$$

$$\underbrace{\varepsilon \dot{x}_0 + \varepsilon^2 \dot{x}_1 + \varepsilon^3 \dot{x}_2 + \dots}_{\varepsilon \dot{x}} +$$

$$\underbrace{x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots}_{x} = 0$$

Collect powers of ε:

$$\boxed{(\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + \dot{x}_0 + x_1) + \varepsilon^2(\ddot{x}_2 + \dot{x}_1 + x_2) + \dots = 0} \quad (1)$$

IC: $x(t=0) = 1$

$$x_0(t=0) + \varepsilon x_1(t=0) + \varepsilon^2 x_2(t=0) + \dots = 1$$

$$\boxed{(x_0(t=0) - 1) + \varepsilon(x_1(t=0)) + \varepsilon^2(x_2(t=0)) + \dots = 0} \quad (2)$$

$$\dot{x}(t=0) = 0$$

$$\boxed{\underline{\dot{x}_0(t=0)} + \varepsilon \dot{x}_1(t=0) + \varepsilon^2 \dot{x}_2(t=0) + \ldots = 0} \quad (3)$$

Now "solve" (1)–(3) by setting terms in increasing powers of ε to zero.

$\underline{O(\varepsilon^0)}$:

$$\boxed{\begin{aligned} \ddot{x}_0 + x_0 &= 0 \\ x_0(t=0) &= 1 \\ \dot{x}_0(t=0) &= 0 \end{aligned}}$$

orig. problem for $\varepsilon = 0$

$\Rightarrow x_0(t) = \cos(t)$

$\underline{O(\varepsilon^1)}$:

$$\boxed{\begin{aligned} \ddot{x}_1 + x_1 &= -\dot{x}_0 \\ x_1(t=0) &= 0 \\ \dot{x}_1(t=0) &= 0 \end{aligned}}$$

$\Rightarrow x_1(t)$

$\underline{O(\varepsilon^2)}$:

$$\boxed{\begin{aligned} \ddot{x}_2 + x_2 &= -\dot{x}_1 \\ x_2(t=0) &= 0 \\ \dot{x}_2(t=0) &= 0 \end{aligned}}$$

$\vdots$

Solve these problems in sequence

$O(\varepsilon^0)$: $\boxed{x_0(t) = \cos(t)}$

(This the "easy" soln. of the orig. problem for $\varepsilon = 0$)

$O(\varepsilon^1)$: $\boxed{\ddot{x}_1 + x_1 = -\dot{x}_0 = \sin(t)}$

$$x_{1H} = A\cos t + B\sin t$$

$$x_{1P} = C\,t\sin t + D\,t\cos t$$

$$\vdots$$

$$x_1 = A\cos t + B\sin t - \frac{1}{2}t\cos t$$

Apply ICs

$$\vdots$$

$$\boxed{x_1(t) = \frac{1}{2}\sin(t) - \frac{1}{2}t\cos t}$$

$O(\varepsilon^2)$: $\quad \ddot{x}_2 + x_2 = -\dot{x}_1 = -\frac{1}{2} t \sin t$

$x_{2H} = A \cos t + B \sin t$

$x_{2P} = -\frac{1}{8} t \sin t + \frac{1}{8} t^2 \cos t$

x_{2P} satisfies I.C. too.

$x_2(t) = x_{2P}(t)$

$\vdots$

$$x(t;\varepsilon) = \cos(t) + \varepsilon\left(\frac{1}{2}\sin(t) - \frac{1}{2} t \cos(t)\right) + \varepsilon^2\left(-\frac{1}{8} t \sin(t) + \frac{1}{8} t^2 \cos(t)\right) + \cdots$$

Mechanical oscillator with weak damping

- Governing (linear) ODE:

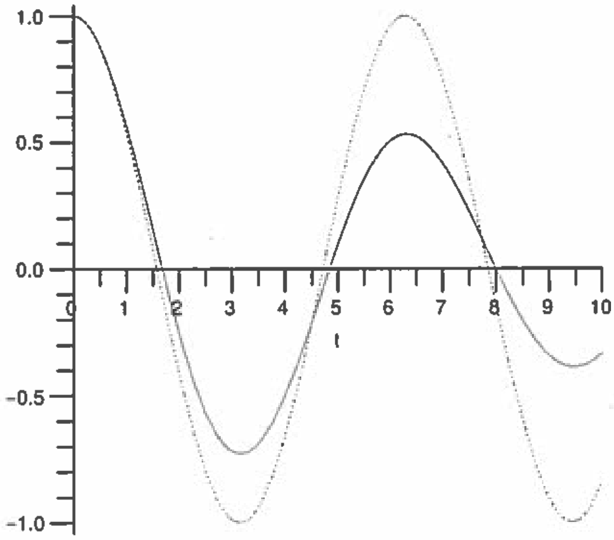
$$\ddot{x} + \epsilon\dot{x} + x = 0$$

subject to the initial conditions

$$x(t=0) = 1 \quad \text{and} \quad \dot{x}(t=0) = 0.$$

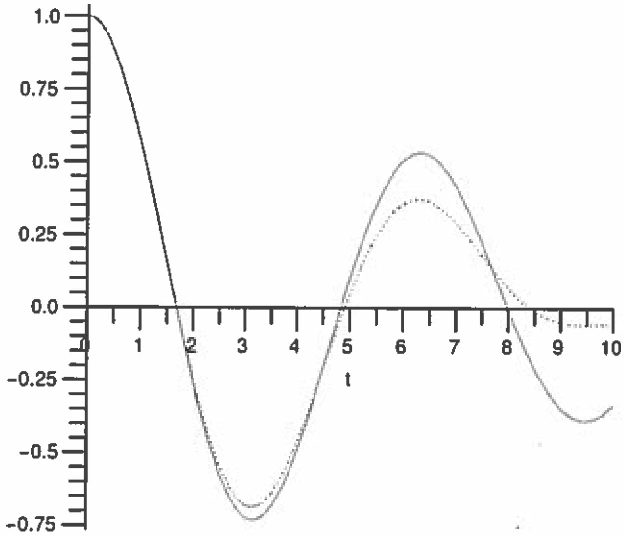
Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- One-term perturbation solution (red), exact solution (green):

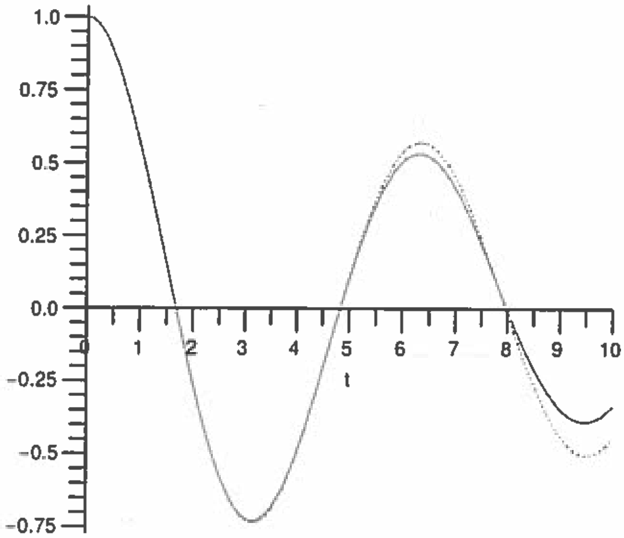


Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Two-term perturbation solution (red), exact solution (green):

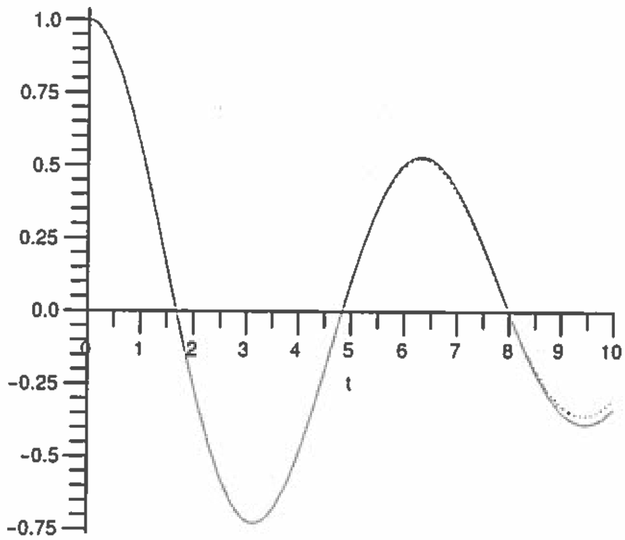


- Three-term perturbation solution (red), exact solution (green):

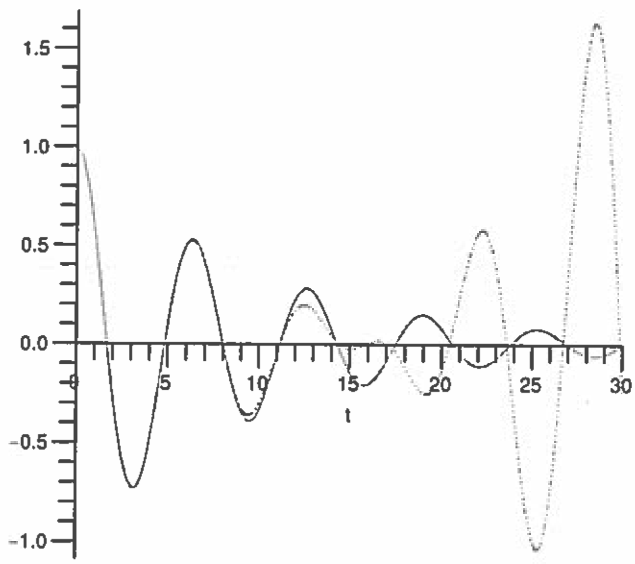


Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Four-term perturbation solution (red), exact solution (green):



- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:



observations:

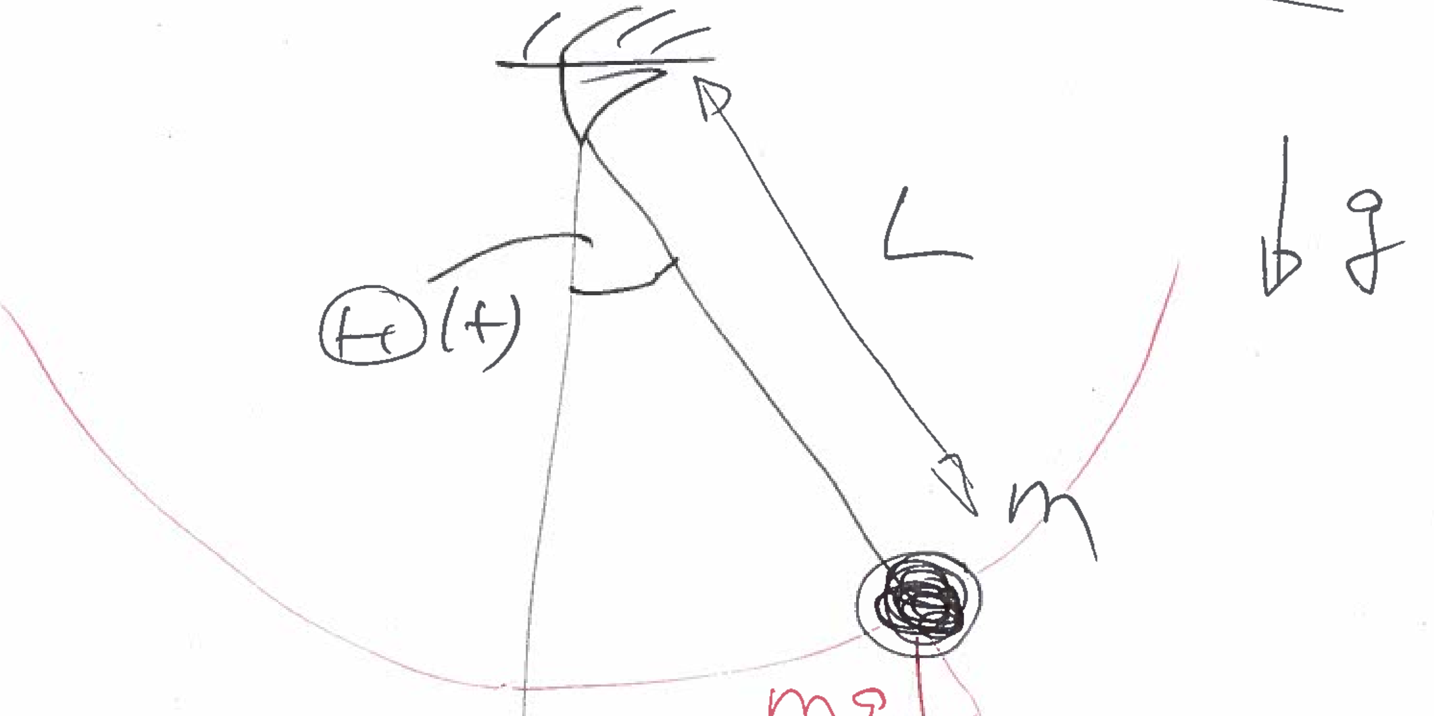
① Perturbation soln. converges rapidly (at fixed time) as

- the number of terms is increased
- ε is decreased (not shown)

② As $t \to \infty$ the solution becomes inaccurate

- because perturbation soln. contains t^n terms with $n > 0$
- errors in the soln. of the ODE "accumulate".

A nonlinear example



Newton's law in tangential direction:

$$\cancel{m}\frac{d}{dt}\Big(\underbrace{L\frac{d\Theta}{dt}}_{\text{tangential veloc}}\Big) = -\cancel{m}g\sin\Theta$$

$$\boxed{\ddot{\Theta} + \omega^2 \sin\Theta = 0 \qquad \omega^2 = \frac{g}{L}}$$

IC: $h(t=0) = \varepsilon$

$$\left.\frac{dh}{dt}\right|_{t=0} = 0$$

IC