Problem has small parameter
Problem is "easy" for \( \varepsilon = 0 \)

\[
x^2 + 3x - 1 = 0
\]

\[
x(\varepsilon) = -\frac{3}{2} \pm \sqrt{\frac{1}{4} 3^2 + 1}
\]

Small \( \varepsilon \): Taylor expand

\[
x(\varepsilon) = \pm 1 - \frac{1}{2} \varepsilon \pm \frac{1}{8} \varepsilon^2 + \ldots
\]

**Ansatz:**

\[
x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots
\]

Into eqn & collect powers of \( \varepsilon \)

\[
(x_0^2 - 1) + \varepsilon (2x_0 x_1 + x_0) + \varepsilon^2 (x_1^2 + 2x_0 x_2 + x_0) + \ldots = 0
\]

\[
x(\varepsilon) = \pm 1 - \frac{1}{2} \varepsilon \pm \frac{1}{8} \varepsilon^2 + \ldots
\]
Note the structure of the eqns:

- Lowest-order eqn is the full eqn for $\varepsilon \approx 0$, which we assumed was "easy".
- Higher-order eqns provide corrections via a hierarchy of problems.
- Eqn. itself is sophisticated up to a quantifiable error.

Questions:

- How do we "guess" the right form of the expansion?
- Will it always work?
An ODE example

Damped oscillator with weak damping:

\[ \ddot{x} + \omega \dot{x} + x = 0 \]

\[ x(t=0) = 1 \]

\[ x_0(t=0) = 0 \]

\[ \Rightarrow x(t; \epsilon) \]

Assume we don't know the form. Note: for \( \epsilon \to 0 \): \( x(t; \epsilon = 0) = \cos t \)

Ansatz for small \( \epsilon \):

\[ x(t; \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots \]

Into problem:

\[ \dot{x}(t) = x(t) + \epsilon \dot{x}_1(t) + \epsilon^2 \dot{x}_2(t) + \ldots \]

\[ \ddot{x}(t) = \ddot{x}_0(t) + \epsilon \ddot{x}_1(t) + \epsilon^2 \ddot{x}_2(t) + \ldots \]

Into ODE
\[ X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots \]
\[ \varepsilon X_0 + \varepsilon^2 X_1 + \varepsilon^3 X_2 + \ldots + \varepsilon X \]
\[ X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots = 0 \]

Collect powers of \( \varepsilon \):
\[ (X_0 + X_0) + \varepsilon (X_1 + X_0 + X_1) + \varepsilon^2 (X_2 + X_1 + X_2) + \ldots = 0 \] (1)

**IC:**
\[ X(t=0) = 1 \]
\[ X_0(t=0) + \varepsilon X_1(t=0) + \varepsilon^2 X_2(t=0) + \ldots = 1 \]
\[ (X_0(t=0) - 1) + \varepsilon (X_1(t=0)) + \varepsilon^2 (X_2(t=0)) + \ldots = 0 \] (2)
\( X(t=0) = 0 \)

\[
\begin{align*}
X_0(t=0) + \varepsilon X_1(t=0) + \varepsilon^2 X_2(t=0) + \ldots & = 0 \\
\varepsilon & = 0 
\end{align*}
\]

(3)

Now "solve" (1) - (3) by setting terms in increasing powers of \( \varepsilon \) to zero.

\( O(\varepsilon^0) \):  
\[
\begin{align*}
X_0 + X_0 & = 0 \\
X_0(t=0) & = 1 \\
X_0(t=\infty) & = 0 
\end{align*}
\]

\( X_0(+) \) = \cos(t)

\( O(\varepsilon^1) \):  
\[
\begin{align*}
X_1 + X_1 & = -X_0 \\
X_1(t=0) & = 0 \\
X_1(t=\infty) & = 0 
\end{align*}
\]

\( X_1(+) \) = \sin(t)

\( O(\varepsilon^2) \):  
\[
\begin{align*}
X_2 + X_2 & = -X_1 \\
X_2(t=0) & = 0 \\
X_2(t=\infty) & = 0 
\end{align*}
\]
Solve these problems in sequence:

\( O(\epsilon^0) \): \[ x_0(t) = \cos(t) \]

(This is the "easy" soln. of the orig. problem for \( \epsilon = 0 \))

\( O(\epsilon^1) \):

\[ x_1 + x_1 = -x_0 = \sin(t) \]

\[ x_{1t} = A \cos t + B \sin t \]
\[ x_{1p} = C t \sin t + D t \cos t \]

\[ x_1(t) = A \cos t + B \sin t - \frac{1}{2} t \cos t \]

Apply ICSs:

\[ x_1(1) = \frac{1}{2} \sin(1) - \frac{1}{2} \cos t \]
0(ε²): \[ x_2 + x_2 = -x_1 = -\frac{1}{2} + \sin t \]

\[ x_{2t} = A \cos t + B \sin t \]

\[ x_p = -\frac{1}{8} t \sin t + \frac{1}{8} t^2 \cos t \]

\[ x_{2p} \text{ satisfies } f(x) = 0. \text{ hoa.} \]

\[ x_2(t) = x_{2p}(t) \]

\[ x(t; \varepsilon) = \cos t + \varepsilon \left( \frac{1}{2} \sin t - \frac{1}{2} + \cos t \right) \]

\[ + \varepsilon^2 \left( -\frac{1}{8} t \sin t + \frac{1}{8} t^2 \cos t \right) \]

+ ...
Mechanical oscillator with weak damping

- Governing (linear) ODE:

\[ \ddot{x} + \varepsilon \dot{x} + x = 0 \]

subject to the initial conditions

\[ x(t = 0) = 1 \quad \text{and} \quad \dot{x}(t = 0) = 0. \]

Comparison between perturbation solution and exact solution for \( \varepsilon = 0.2 \)

- One-term perturbation solution (red), exact solution (green):
Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Two-term perturbation solution (red), exact solution (green):

- Three-term perturbation solution (red), exact solution (green):
Comparison between perturbation solution and exact solution for $\epsilon = 0.2$

- Four-term perturbation solution (red), exact solution (green):

- Agreement over a finite time-interval is very pleasing. However, over sufficiently large times, the perturbation solution diverges:
observations:

① Perturbation $S_{\text{en}}$ converges rapidly (at fixed time) as:
   - the number of terms is increased
   - $\varepsilon$ is decreased (not shown)

② As $t \to \infty$, the solution becomes inaccurate:
   - because perturbation $S_{\text{en}}$ contains $t^n$ terms with $n > 0$
   - errors in the solution of the ODE accumulate.
A non-linear example

Newton's law in tangential direction:

\[
m \frac{d}{dt} \left( L \frac{d\Theta}{dt} \right) = -mg \sin\Theta
\]

tangential velocity

\[
\ddot{\Theta} + \omega^2 \sin\Theta = 0
\]

\[
\omega^2 = \frac{g}{L}
\]
\[ \text{CE: } \theta(t) = 3 \]

\[ \frac{d\theta}{dt} \bigg|_{t=0} = 0 \]