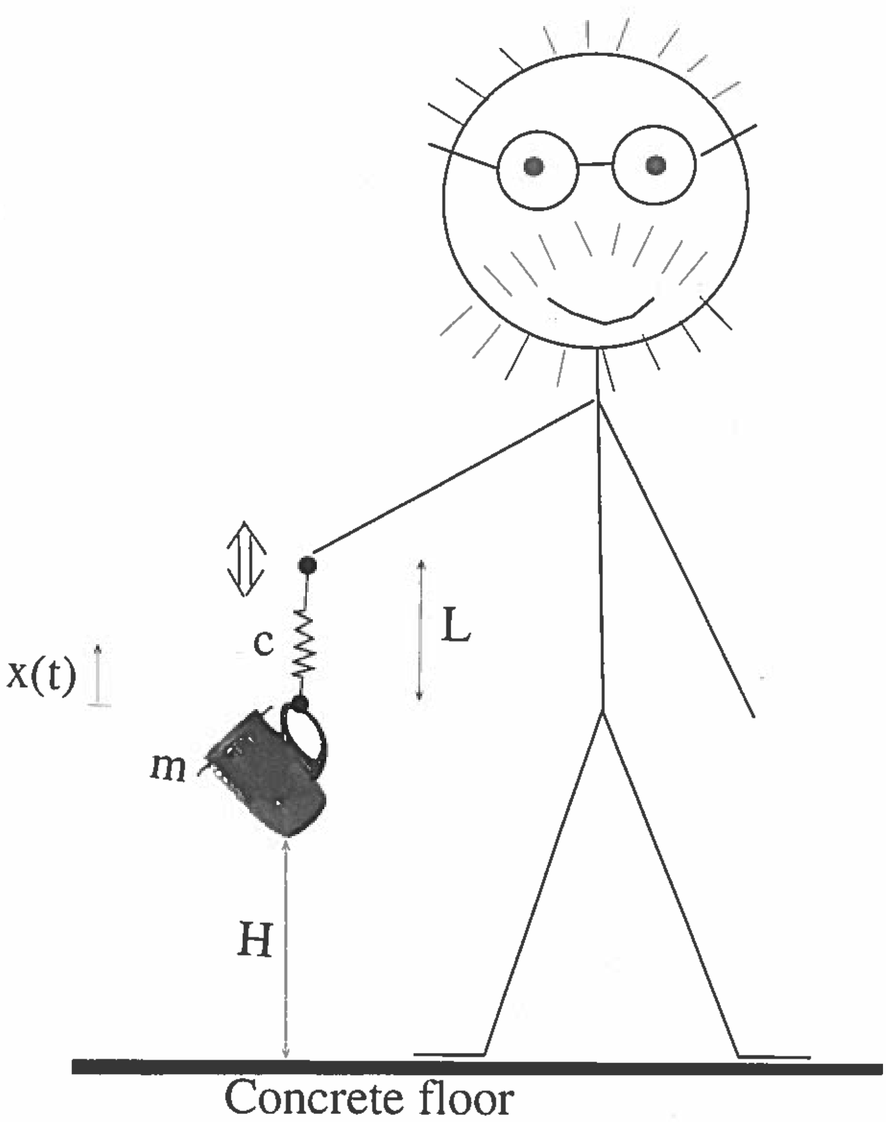
The experiment



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$$m\ddot{x} + k\dot{x} + cx = \hat{F}\cos(\Omega t)$$

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = \hat{f}\cos\Omega t$$

$$= Re(\hat{f}e^{i\Omega t})$$

$$x(t) = x_p + x_h$$

↳ to zero as $t \to \infty$

$$x_p(t) = Re(X e^{i\Omega t})$$

$$\frac{|X|}{(\hat{f}/\omega^2)} = \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2 + \left(2\left(\frac{\delta}{\omega}\right)\left(\frac{\Omega}{\omega}\right)\right)^2}}$$

what if $\delta = 0$ (no damping?)

& $\Omega = \omega \Rightarrow$ ~~Re~~ Resonance

If $\delta = 0$:

$$\ddot{x} + \omega^2 x = \hat{f}\cos(\Omega t) = \mathrm{Re}\left(\hat{f} e^{i\Omega t}\right)$$

As above unless $\Omega = \omega$.
In that case: $\hat{f}\cos(\Omega t)$ is a solution of the homog. ODE. Remedy

Ansatz:

$$x_p = C' t\cos(\Omega t) = \mathrm{Re}\left(C' t e^{i\Omega t}\right)$$

$$\ddot{x}_p = \mathrm{Re}\left(C' e^{i\Omega t}\left(2i\Omega - \Omega^2 t\right)\right)$$

into ODE:

$$C' \cancel{e^{i\Omega t}}\left(\underbrace{2i\Omega - \cancel{\Omega^2 t}}_{\ddot{x}} + \underbrace{\cancel{\Omega^2 t}}_{x}\right) \overset{!}{=} \hat{f}\cancel{e^{i\Omega t}}$$

$$C' = \frac{\hat{f}}{2i\Omega} = -i\frac{\hat{f}}{2\Omega}$$

$$x_p = \mathrm{Re}\left(-i\frac{\hat{f}}{2\Omega} t \underbrace{e^{i\Omega t}}_{\left(\cos(\Omega t) + i\sin(\Omega t)\right)}\right)$$

$$x_p = \frac{\hat{f}}{2\Omega} t \sin(\Omega t)$$

Complete solution:

$$x(t) = \underbrace{A\cos(\Omega t) + B\sin(\Omega t)}_{x_H} + \underbrace{\frac{\hat{f}}{2\Omega} t \sin(\Omega t)}_{x_p}$$

$x_p \rightarrow \infty$ as $t \rightarrow \infty$

mug hits the floor!

Basic ideas of perturbation methods: "Exploiting small parameters" and "Scaling"

Observation 1:

- ODEs (and hence their solutions!) typically contain some parameters, e.g.

  $$m\ddot{x} + k\dot{x} + cx = F\ \cos(\Omega t)$$

  so

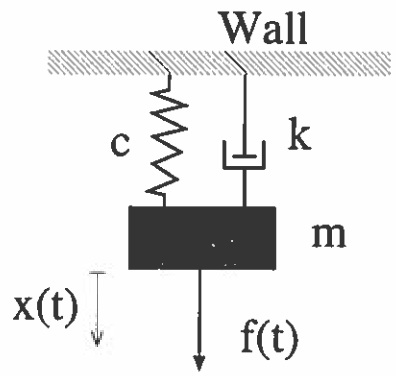
  $$x = x(t) = x(t; m, k, c, \Omega).$$

- Often some of the problem's parameters are "small". How can we exploit this?

- Example:
  - Assume that we (only) know the solution of the above ODE for $k = 0$ (no damping).
  - What is the solution for "small" k?

Observation 2:

- ODEs that model physical phenomena typically express balances (of forces, energies, currents, ...).
- Here's an example of a balance of forces:



$$\underbrace{m\ddot{x}}_{\text{inertial forces}} + \underbrace{k\dot{x}}_{\text{damping forces}} + \underbrace{cx}_{\text{spring forces}} = \underbrace{F\cos(\Omega t)}_{\text{applied external force}}$$

- In general, all terms in the ODE will make a significant contribution to the overall "balance".
- However, there *may* be regimes in which the balance of terms is dominated by a balance between just a few (ideally two) terms, while the other terms only provide "negligible" contributions.
- The simplified equations (obtained by neglecting the small terms) are often much easier to solve than the full equations.
- We may [should!] then be interested in finding the effect that the "small" perturbations have on the solution.
- A seemingly trivial observation: You will need *at least* two terms to balance!

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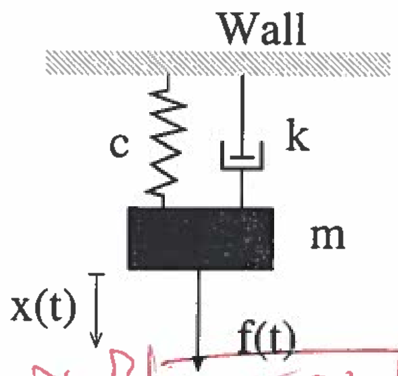
Example:



$\sim\Omega^2 \qquad \sim\Omega \qquad \sim 1$

$$m\ddot{x} + k\dot{x} + cx = F\cos(\Omega t)$$

- We established earlier that

$$x(t) = \boxed{x_P(t)} + x_H(t)$$

$\to 0$ as $t \to \infty$

where $x_H(t) \to 0$ very rapidly.

- Following the decay of the initial transients [described by $x_H(t)$] we have

$$x(t) \approx x_P(t) = A\cos(\Omega t) + B\sin(\Omega t)$$

$\dot{x} \sim \Omega(-A\sin(\Omega t) + B\cos(\Omega t))$

- Hence if Ω is "small", the mass will move very slowly, implying that $m\ddot{x}$ and $k\dot{x}$ will be much smaller than cx.

$\ddot{x} \sim \Omega^2$

- In this "quasi-steady" regime, we expect the motion of the mass to be described (approximately!) by

$$cx(t) \approx F\cos(\Omega t).$$

$x(t) \simeq \frac{F}{c}\cos\Omega t$

"Proof"

- Check that
$$x(t) \approx \frac{F}{c}\cos(\Omega t)$$
is an approximate solution of
$$m\ddot{x} + k\dot{x} + cx = F\cos(\Omega t)$$
if Ω is small.

- The exact solution is
$$x(t) \approx x_P(t) = A\cos(\Omega t) + B\sin(\Omega t)$$
where
$$A = F\frac{c - \cancel{m\Omega^2}}{\cancel{(k\Omega)^2} + (c - \cancel{m\Omega^2})^2} \quad \to \frac{F}{c} \quad \text{as } \Omega \to 0,$$
and
$$B = F\frac{k\Omega}{\cancel{(k\Omega)^2} + (c - \cancel{m\Omega^2})^2} \quad \to 0 \quad \text{as } \Omega \to 0.$$

"Q.E.D."

Observations about Observations 1, 2 and 3

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- The approach outlined above exploits *additional* knowledge about the problem.
- You will either have such knowledge *a priori* or you can make certain (hopefully plausible) assumptions about certain properties of the solution.
- In the latter case, you'll have to check the consistency of your assumptions when you're done. For instance:
  - Assume the the solution is such that certain terms in the ODE are small.
  - Neglect the small terms in the ODE and solve.
  - Check afterwards that the terms that were *assumed* to be small are *actually* small.
- The approach tends to produce approximate solutions of the ODE that are valid only in certain *"regions of parameter space"*, e.g. for small forcing frequencies Ω, small damping k, etc.
- This is often more useful than having an exact (but horrendously complicated) closed-form solution that is valid for all parameter values.

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Exploiting small parameters:

(Regular) Perturbation expansions

An algebraic example:

$$\boxed{x^2 + \varepsilon x - 1 = 0}$$

$\underline{\varepsilon = 0}$:

$x^2 = 1$

$x = \pm 1$

$$x = -\frac{1}{2}\varepsilon \pm \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + 1}$$

$$x(\varepsilon) = -\tfrac{1}{2}\varepsilon \pm \left(\frac{1}{4}\varepsilon^2 + 1\right)^{1/2}$$

Note: for small positive ε we can approximate $x(\varepsilon)$ by its Taylor series

$$x(\varepsilon) = \underline{x(0)} + \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + \frac{1}{2}\left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0} \varepsilon^2 + \ldots$$

$$x(0) = \pm 1$$

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{1}{2} \; ; \; \left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0} = \pm\frac{1}{4}$$

$$x(\varepsilon) = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 \mp \frac{1}{128}\varepsilon^3 + \dots$$

Power series representation in ε. Radius of convergence: Series converges for $|\varepsilon| < 2$.
But more importantly the series gives a good approx. to the exact soln. for small ε.

Observations:

Have solution $\rightarrow$ expand in powers of ε

Idea:

Pose the solution in the form of a power series $\rightarrow$ Ansatz!

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

Into eqn: $x^2 + \varepsilon x - 1 = 0$

$$(x_0+\varepsilon x_1+\varepsilon^2 x_2+\dots)^2+$$

$$\varepsilon(x_0+\varepsilon x_1+\varepsilon^2 x_2+\dots)-1=0$$

Expand & collect powers of ε:

~~1st term~~ 1st term:

$$(x_0+\varepsilon x_1+\varepsilon^2 x_2+\dots)(x_0+\varepsilon x_1+\varepsilon^2 x_2+\dots)$$

$$=x_0^2+\varepsilon(2x_0x_1)+\varepsilon^2(2x_0x_2+x_1^2)$$

$$+\dots$$

~~Collect~~

Collect powers of ε:

$$\underline{x_0^2}+\varepsilon(2x_0x_1)+\varepsilon^2(2x_0x_2+x_1^2)+\dots$$

$$+\varepsilon x_0+\varepsilon^2 x_1+\varepsilon^3 x_2+\dots\underline{-1}=0$$

$$(x_0^2-1)\varepsilon^0+(2x_0x_1+x_0)\varepsilon^1+$$

$$+(2x_0x_2+x_1^2+x_1)\varepsilon^2+\dots=0$$

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<u>Idea:</u> want the LHS to be zero, so set the coefficients of increasing powers of ε to zero!

$\underline{\varepsilon^0:}$ $x_0^2 - 1 = 0 \Rightarrow x_0 = \pm 1$

$\underline{\varepsilon^1:}$ $2x_0x_1 + x_0 = 0$

$x_0(2x_1 + 1) = 0 \Rightarrow x_1 = -\frac{1}{2}$

$\underline{\varepsilon^2:}$ $2x_0x_2 + x_1^2 + x_1 = 0 \Rightarrow x_2 = \pm\frac{1}{8}$

etc.

<u>So:</u>

$$x(\varepsilon) = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \dots$$

As before

Note the structure of the soln:

- Lowest-order eqn is equal to the full eqn for $\varepsilon = 0$ & we had assumed that this eqn was "easy" to solve.
- Higher-order eqns provide systematic corrections to the solution via a hierarchy of eqns.
- Eqn itself is satisfied to an increasing accuracy the more terms are included.

Looks promising!

Issues: How do we guess the right form of the expansion?