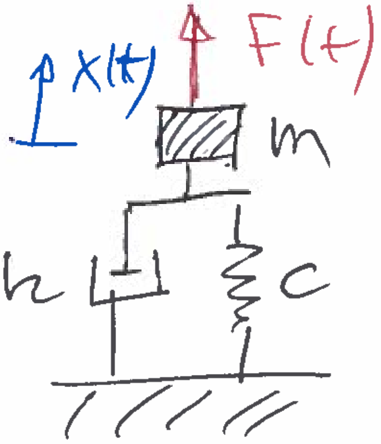


$$m\ddot{x} + h\dot{x} + cx = F(t)$$

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = f(t)$$

$$\delta = \frac{h}{2m} \qquad \omega^2 = \frac{c}{m} \qquad f(t) = \frac{F(t)}{m}$$

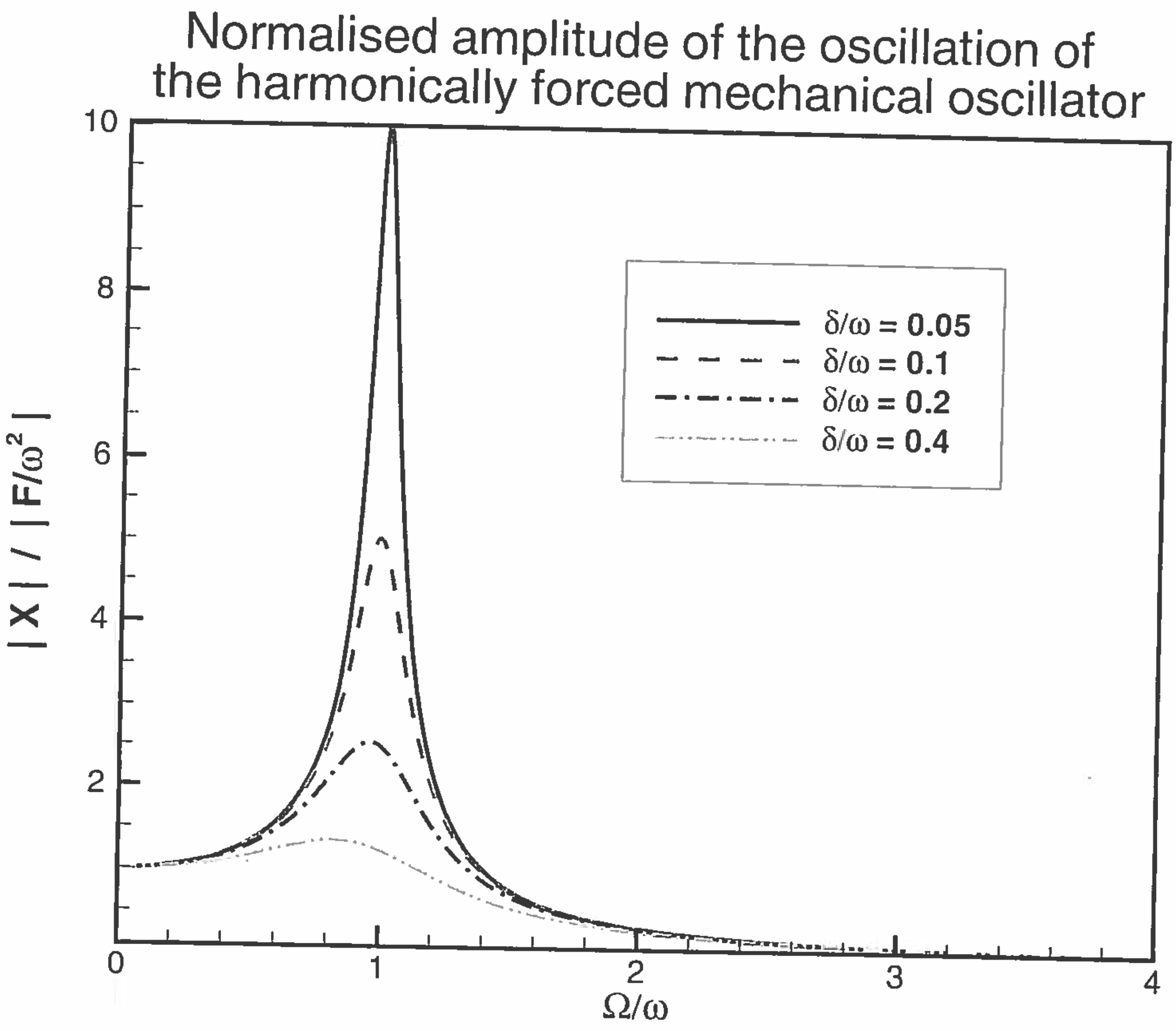
$$x(t) = \underbrace{x_h(t)}_{\text{free oscillations}} + \underbrace{x_p(t)}_{\text{forced oscillations}}$$

$$x_p(t) = X e^{i\Omega t} \quad \text{for } f(t) = \hat{f} e^{i\Omega t}$$

$$\frac{|X|}{\left(\hat{f}/\omega^2\right)} = \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2 + \left(2\left(\frac{\delta}{\omega}\right)\left(\frac{\Omega}{\omega}\right)\right)^2}}$$

where $\frac{\hat{f}}{\omega^2} = \frac{\hat{F}}{c}$

see plot: $|X|$ has a max. at $\Omega = \omega$



Normalised amplitude of the oscillation of
the harmonically forced mechanical oscillator
| X | / | F/ω² |
Ω/ω
δ/ω = 0.05
δ/ω = 0.1
δ/ω = 0.2
δ/ω = 0.4
0
1
2
3
4
2
4
6
8
10

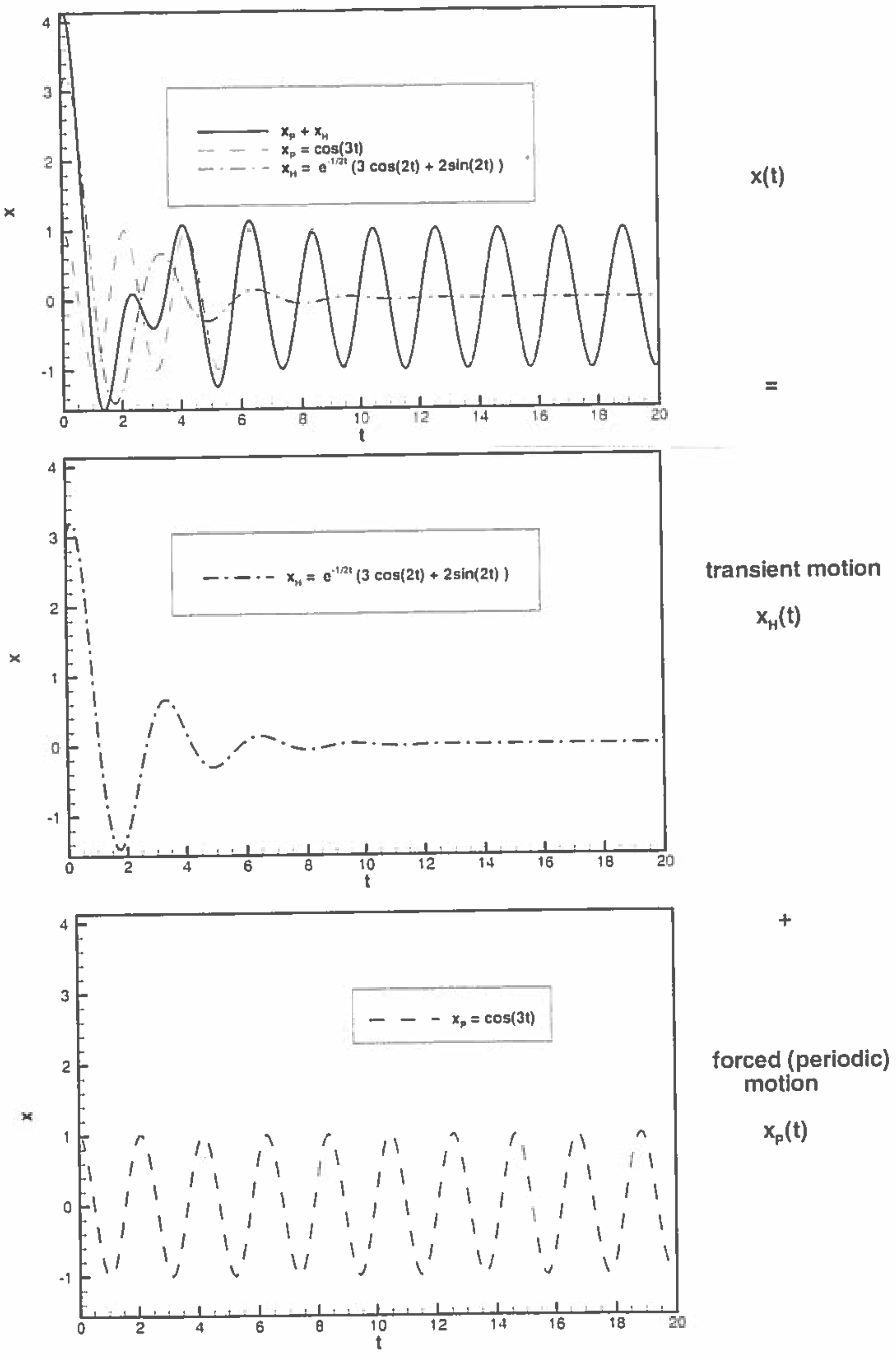


Figure 5: The displacement of a harmonically-forced, damped mechanical oscillator comprises the periodic (forced) solution $x_P(t)$ and the transient solution $x_H(t)$.

but $|X|$ stays finite for $\delta > 0$. (Always true in real systems) 2

Finally consider the case $\delta = 0$ (<u>resonance</u>)

In that case:

$$\ddot{x} + \omega^2 x = \hat{f} e^{i\Omega t}$$

As before, unless $\Omega = \omega$; in that case $r(t) = f(t) \sim e^{i\Omega t}$ is a soln. of the homog. ODE

modified ansatz:

$$x_p = C\, t\, e^{i\Omega t}$$

$$\ddot{x}_p = C e^{i\Omega t} (2i\Omega - \Omega^2 t)$$

into ODE:

$$C \cancel{e^{i\Omega t}} \Big(\underbrace{2i\Omega - \cancel{\Omega^2 t}}_{\ddot{x}} + \underbrace{\cancel{\Omega^2 t}}_{\omega^2 X} \Big) \overset{!}{=} \hat{f} \cancel{e^{i\Omega t}}$$

$$C = \frac{\hat{f}}{2i\Omega} = -i\frac{\hat{f}}{2\Omega}$$

$$x_p(t) = -i\frac{\hat{f}}{2\Omega}te^{i\Omega t}$$

Now extract real (or imag) part for $f = \hat{f}\begin{Bmatrix}\cos(\Omega t)\\ \sin(\Omega t)\end{Bmatrix}$

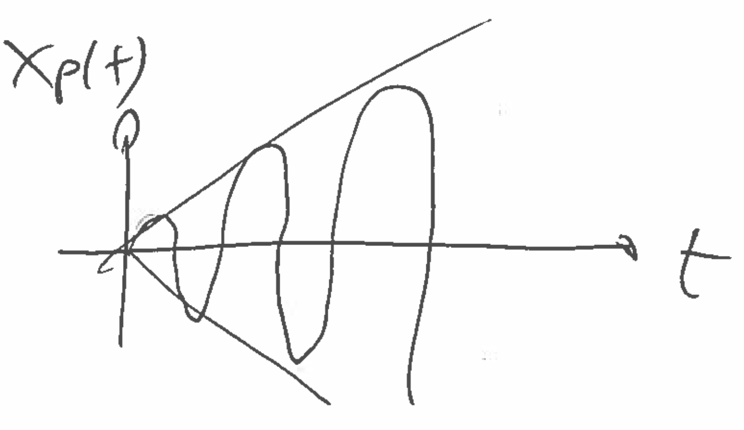
So for real part:

$$x_p(t) = Re\left(-i\frac{\hat{f}}{2\Omega}t\left(\cos(\Omega t) + i\sin(\Omega t)\right)\right)$$

$$= \frac{\hat{f}}{2\Omega}t\sin(\Omega t)$$

So full, general soln:

$$x(t) = A\sin(\Omega t) + B\cos(\Omega t) + \underbrace{\frac{\hat{f}}{2\Omega}t\sin(\Omega t)}$$



Basic ideas of perturbation methods: "Exploiting small parameters" and "Scaling"

Observation 1:

- ODEs (and hence their solutions!) typically contain some parameters, e.g.

$$m\ddot{x} + k\dot{x} + cx = F\ \cos(\Omega t)$$

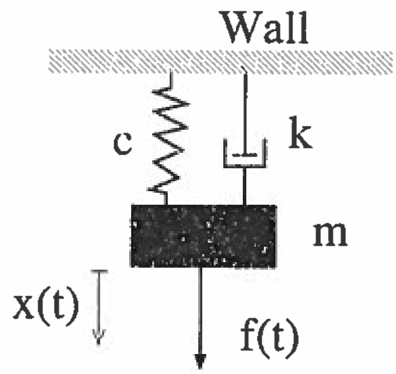
so

$$x = x(t) = x(t; m, k, c, \Omega).$$

- Often some of the problem's parameters are "small". How can we exploit this?
- Example:
  - Assume that we (only) know the solution of the above ODE for $k = 0$ (no damping).
  - What is the solution for "small" k?

Observation 2:

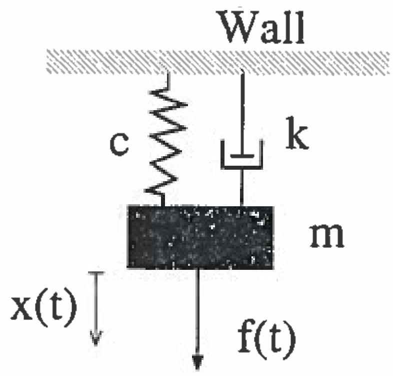
- ODEs that model physical phenomena typically express balances (of forces, energies, currents, ...).
- Here's an example of a balance of forces:



$$\underbrace{m\ddot{x}}_{\text{inertial forces}} + \underbrace{k\dot{x}}_{\text{damping forces}} + \underbrace{cx}_{\text{spring forces}} = \underbrace{F\cos(\Omega t)}_{\text{applied external force}}$$

- In general, all terms in the ODE will make a significant contribution to the overall "balance".
- However, there *may* be regimes in which the balance of terms is dominated by a balance between just a few (ideally two) terms, while the other terms only provide "negligible" contributions.
- The simplified equations (obtained by neglecting the small terms) are often much easier to solve than the full equations.
- We may [should!] then be interested in finding the effect that the "small" perturbations have on the solution.
- A seemingly trivial observation: You will need *at least* two terms to balance!

Example:



$$m\ddot{x} + k\dot{x} + \boxed{cx = F\ \cos(\Omega t)}$$

- We established earlier that

$$x(t) = x_P(t) + \cancel{x_H(t)}$$

  where $x_H(t) \to 0$ very rapidly.

- Following the decay of the initial transients [described by $x_H(t)$] we have

$$x(t) \approx x_P(t) = A\ \cos(\Omega t) + B\ \sin(\Omega t)$$

- Hence if Ω is "small", the mass will move very slowly, implying that $m\ddot{x}$ and $k\dot{x}$ will be much smaller than cx.

- In this "quasi-steady" regime, we expect the motion of the mass to be described (approximately!) by

$$c\,x(t) \approx F\ \cos(\Omega t).$$

$x(t) \approx \frac{F}{c}\cos(\Omega t)$ for $\Omega \ll 1$

"Proof"

- Check that

$$\boxed{x(t) \approx \frac{F}{c}\cos(\Omega t)}$$

is an approximate solution of

$$m\ddot{x} + k\dot{x} + cx = F\cos(\Omega t)$$

if Ω is small.

- The exact solution is

$$x(t) \approx \boxed{x_P(t) = A\cos(\Omega t)} + B\sin(\Omega t)$$

where

$$A = F\frac{c - m\Omega^2}{(k\Omega)^2 + (\boxed{c} - m\Omega^2)^2} \to \boxed{\frac{F}{c} \quad \text{as } \Omega \to 0,}$$

and

$$B = F\frac{k\Omega}{(k\Omega)^2 + (c - m\Omega^2)^2} \to 0 \quad \text{as } \Omega \to 0.$$

"Q.E.D."

§ 6 Perturbation methods

(maybe)

Exploiting small parameters
⟹ (Regular) perturbation expansions

An ~~algeb~~ algebraic example:

$$x^2 + \varepsilon x - 1 = 0$$

$$x = -\frac{1}{2}\varepsilon \pm \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + 1}$$

$$x(\varepsilon) = -\frac{1}{2}\varepsilon \pm \left(\frac{1}{4}\varepsilon^2 + 1\right)^{1/2}$$

__Note:__ If $\varepsilon = 0$ then the soln. becomes much easier! $x = \pm 1$

For small but finite values of ε we can use the Taylor expansion of $x(\varepsilon)$ about $\varepsilon = 0$.

$$x(\varepsilon) = x(0) + \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + \frac{1}{2!} \left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0} \varepsilon^2 + \dots$$

$\vdots$

$$x(0) = \pm 1$$

$$\frac{dx}{d\varepsilon} = -\frac{1}{2} \pm \frac{1}{2}\left(\frac{1}{4}\varepsilon^2+1\right)^{-1/2} \frac{1}{2}\varepsilon$$

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{1}{2}$$

$$\frac{d^2x}{d\varepsilon^2} = \dots$$

$$\left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0} = \pm\frac{1}{4}$$

$$x(\varepsilon) = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \dots$$

This is a good approx. to the exact soln. for small ε.

Observation:

Have the solution → Taylor expand for small ε
→ power series in ε.

Idea:

we don't have the soln so we pose the soln. as a power series → ansatz:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

Task: determine the coefficients $x_0, x_1, x_2, \dots$

Insert ansatz into eqn: (4)

$$x^2 + \varepsilon x - 1 = 0$$

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 +$$
$$\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) - 1 = 0$$

Collect powers of ε:

$$\underline{x_0^2} + 2\varepsilon x_1 x_0 + \varepsilon^2 2x_2x_0 + \varepsilon^2 x_1^2 + \dots$$
$$+ \varepsilon x_0 + \varepsilon^2 x_1 + \varepsilon^3 x_2 + \dots \underline{-1} = 0$$

$$\boxed{\underline{(x_0^2 - 1)} + \varepsilon(2x_1x_0 + x_0) + \varepsilon^2(2x_2x_0 + x_1^2 + x_1) + \dots = 0}$$

$\boxed{\text{ASIDE}}$

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 \dots)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)$$

5

Set the coefficients of ε to zero, starting with smallest power of ε!

ε^0: $x_0^2 - 1 = 0 \Rightarrow x_0 = \pm 1$

ε^1: $2x_1 x_0 + x_0 = 0$

$x_0(2x_1 + 1) = 0 \Rightarrow x_1 = -\frac{1}{2}$

ε^2: $2x_2 x_0 + x_1^2 + x_1 = 0$

$x_2 = \frac{1}{2x_0}(-x_1 - x_1^2)$

$x_2 = \pm \frac{1}{8}$

etc.

$$x(\varepsilon) \approx x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

$$\approx \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \dots$$