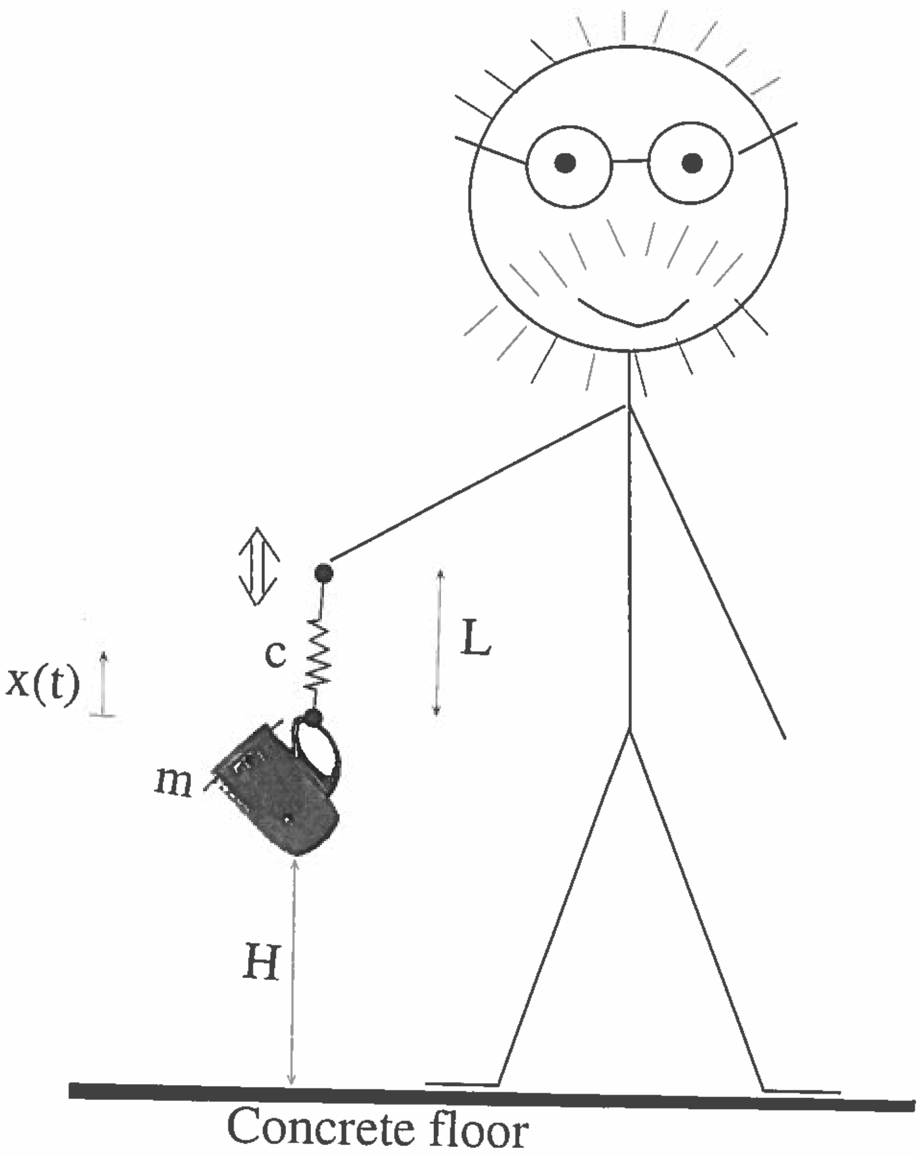
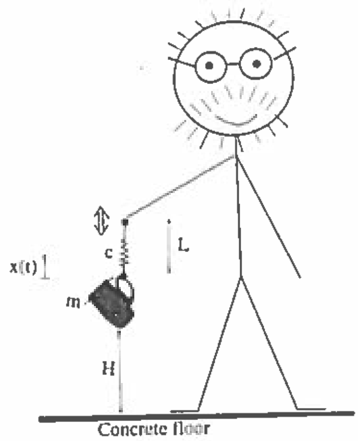
The experiment

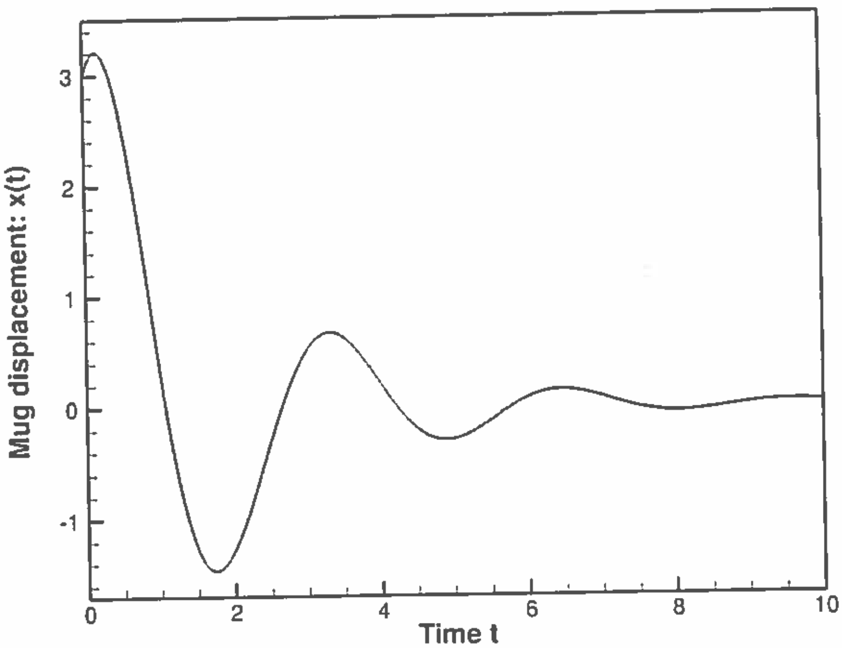


Experiment 1: Free oscillations



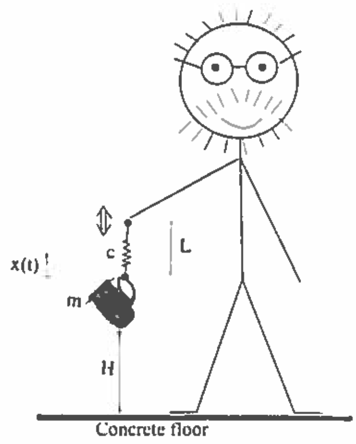
Procedure:

- Deflect mug from its rest position.
- "Let go" and observe the mug's time-dependent motion while keeping hand still.



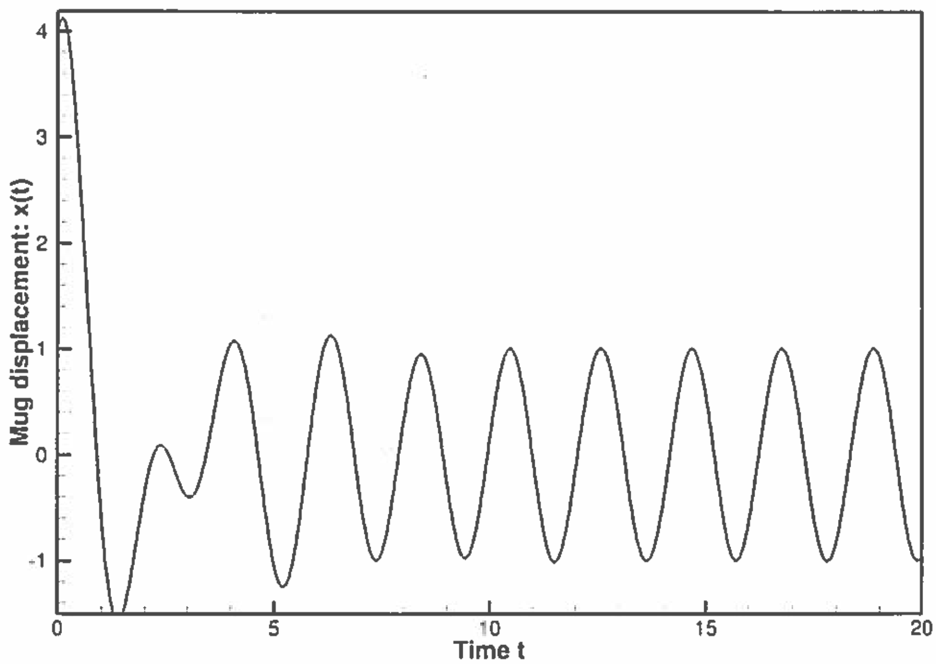
- Damped oscillation with certain characteristic frequency – the system's "cigen" frequency.

Experiment 2: Forced oscillations



Procedure:

- Start from rest.
- Perform time-harmonic oscillations with hand and observe the mug's time-dependent motion.



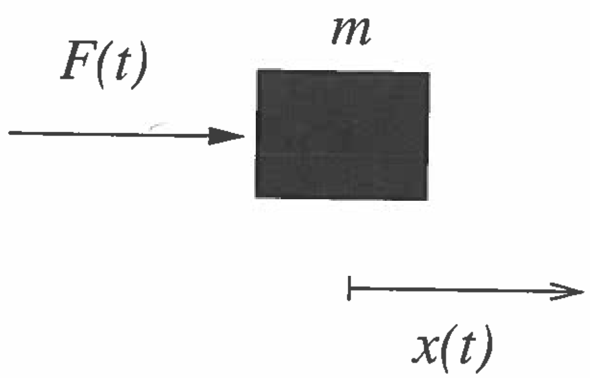
- "Initial transients", followed by time-harmonic "response" with same frequency as "forcing".
- Note: Amplitude of "response" depends on forcing frequency:
  - tiny for high frequencies,
  - same as forcing amplitude for very low frequencies,
  - very large for frequencies near the system's "eigen" frequency.

Everything you always wanted to know about mechanical oscillators but were afraid to ask

- The first half of MATH10222 is not directly concerned with mechanics.
- However, mechanical systems provide nice illustrations of many of the phenomena that we have discussed (or will discuss) in a more abstract mathematical setting.

I. Newton's law for one-dimensional motion

- In words: "*The sum of all forces acting on a particle of mass m is equal to its mass times its acceleration*"

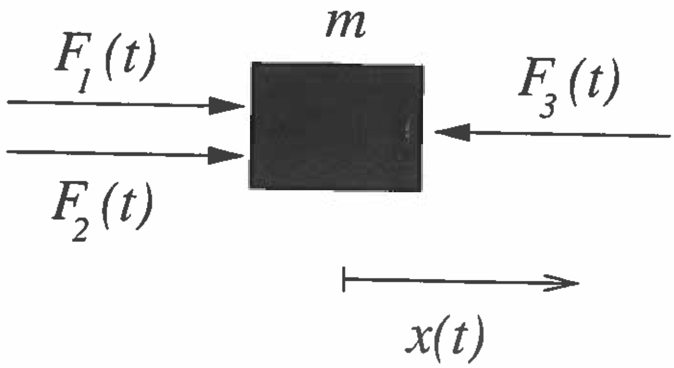


- Or, written as an equation:

$$m\frac{d^2x}{dt^2} = F(t)$$

I. Newton's law for one-dimensional motion (cont.)

- Here's an example with multiple forces



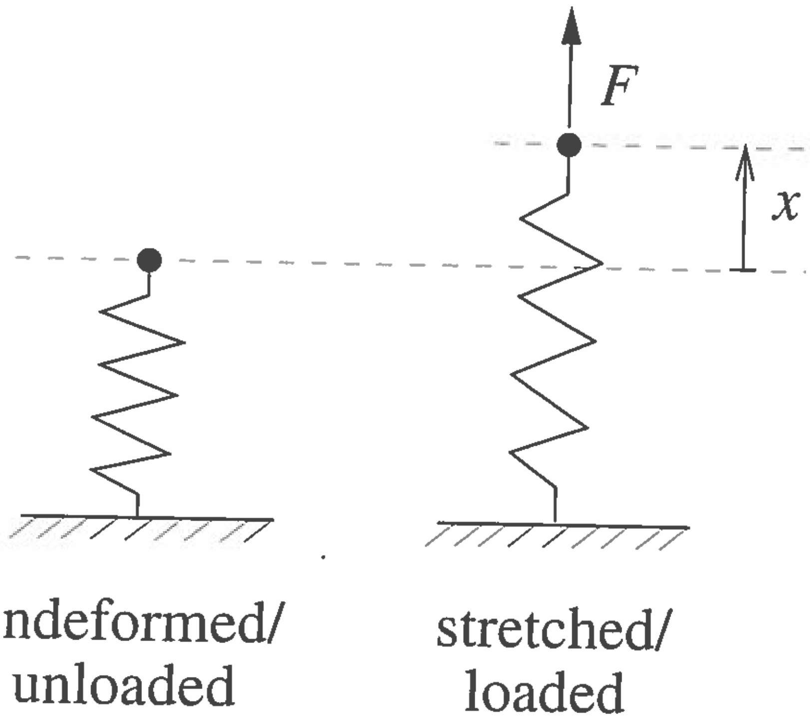
- In this case Newton's law becomes:

$$m\frac{d^2x}{dt^2} = F_1(t) + F_2(t) - F_3(t)$$

- Note the direction of the forces!

II. (Linearly) elastic springs

- Observation: When a spring is loaded by a force, F, its length increases by a certain amount, x, say.



- For a linearly elastic spring we have

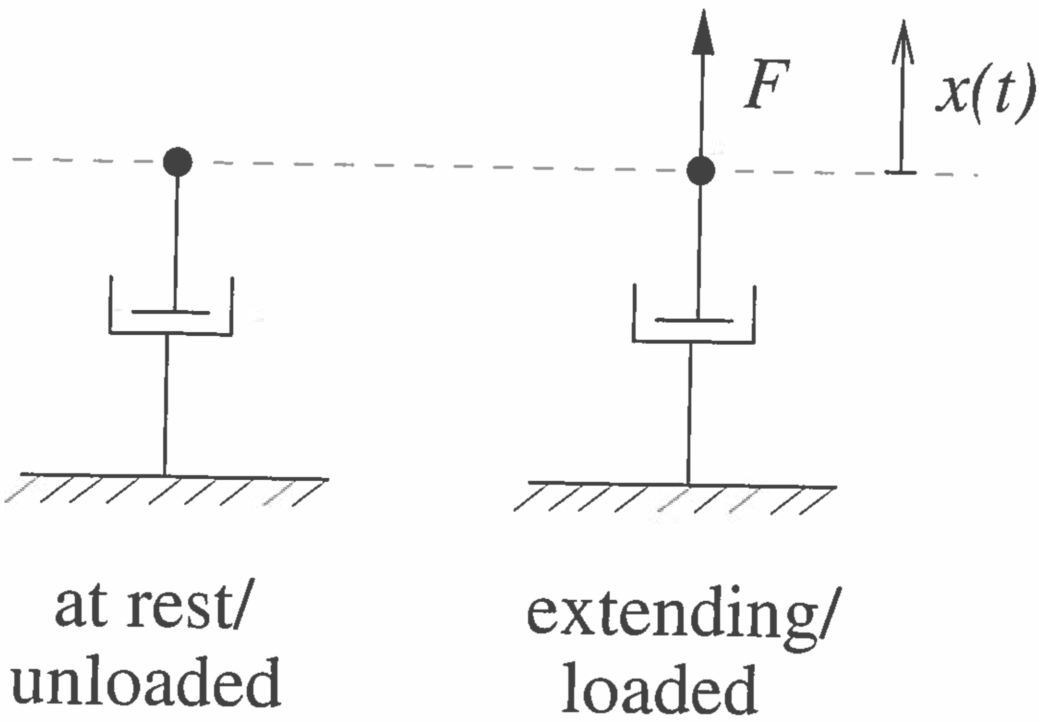
$$F = c\,x$$

where c is the "spring constant", a measure of its stiffness.

- Thus c indicates how strongly the spring resists its *static* extension.

III. (Linear) dampers

- Observation: When a damper is loaded by a force F its length increases at a rate dx/dt:



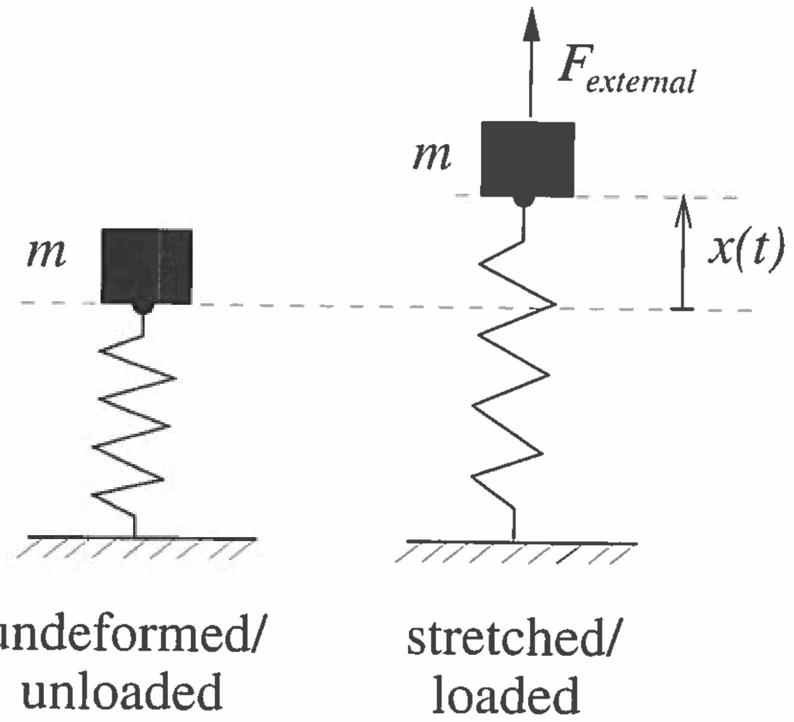
- For a linear damper we have

$$F = k\,\frac{dx}{dt}$$

where k is the "damping constant", a measure of how strongly the damper resists its *dynamic* extension.

IV. Putting it all together: "Action = Reaction"

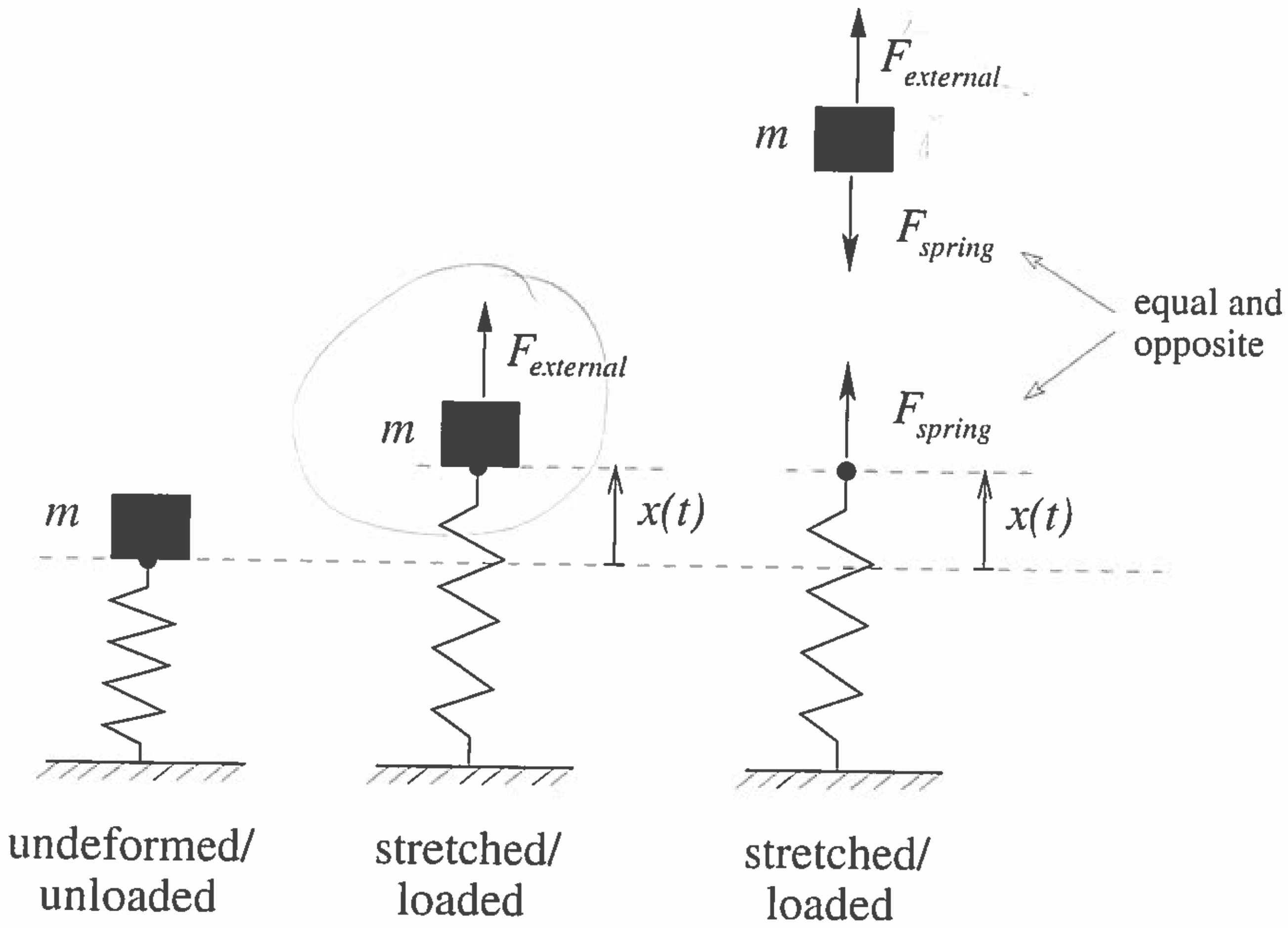
- Here is a mass m, attached to a spring of stiffness c, and loaded by a force, $F_{external}$.



- What is the equation of motion for the mass?
- Write down Newton's law for the mass.
- $\Longrightarrow$ What forces act on the mass?

IV. Putting it all together (cont.)

- "Action = Reaction": The spring pulls the mass and mass pulls the spring (in the opposite direction, obviously!):



- Thus Newton's law states

$$m\frac{d^2x}{dt^2} = F_{external} - F_{spring},$$

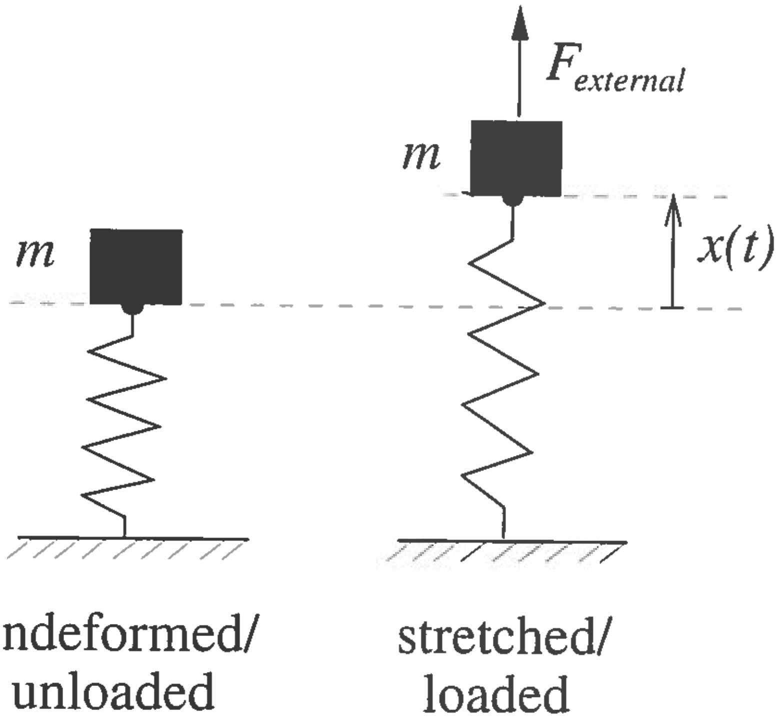
or, using what we've just learned about linear springs:

$$m\frac{d^2x}{dt^2} = F_{external} - cx.$$

IV. Putting it all together (cont.)

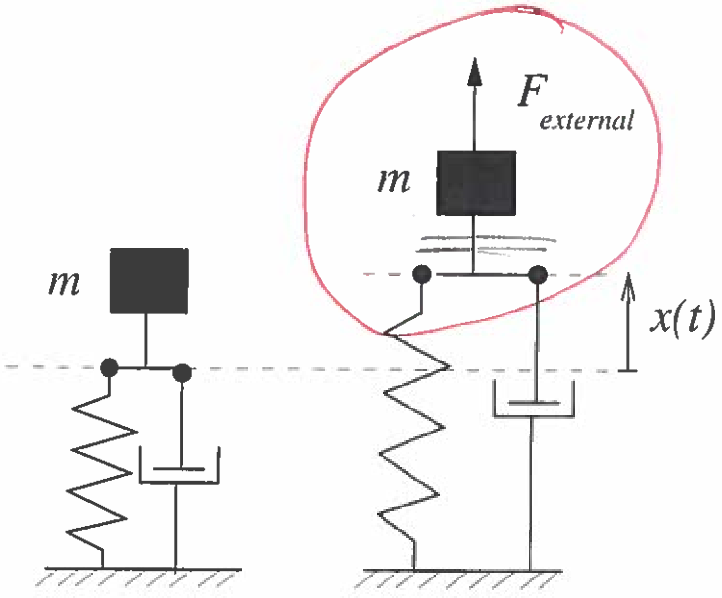
- Rewrite to the standard form of a second-order ODE for $x(t)$:

$$m\frac{d^2x}{dt^2} + cx = F_{external}.$$



Exercise: Try it for yourself

- Here is a mass m, attached to a spring of stiffness c, and a damper (damping constant k), loaded by a force $F_{external}$.

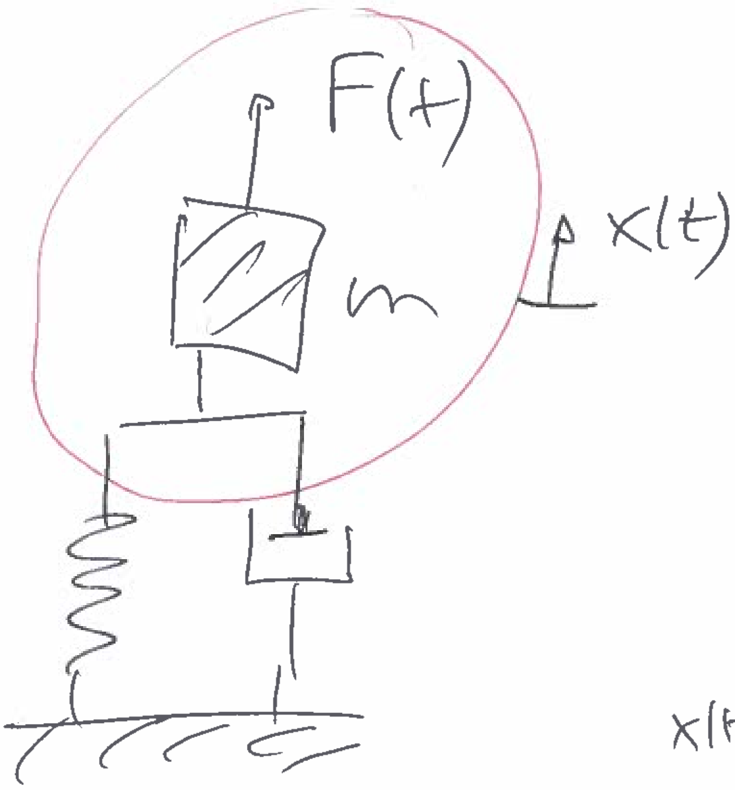


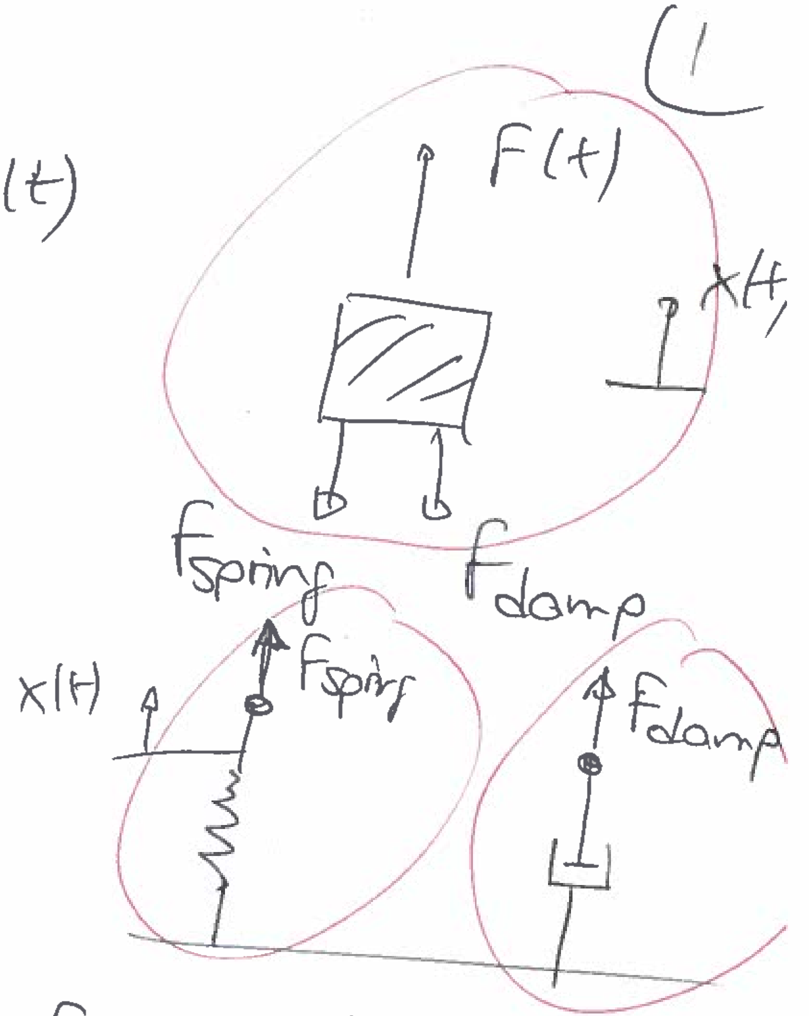
undeformed/ at rest/ unloaded

stretched/ extending/ loaded

- Show that the equation of motion for the mass is

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = F_{external}$$





Newton's law:

$$m\frac{d^2x}{dt^2} = F(t) - f_{spring} - f_{damp}$$

$$f_{spring} = cx \qquad f_{damp} = k\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} = F(t) - c\,x(t) - k\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = F(t)$$

2nd order constant coeff fn. ODE.
for $x(t)$

rewrite in standard form:

$$\ddot{x} + 2\delta \dot{x} + \omega^2 x = f(t)$$

where

$$\delta = \frac{k}{2m} > 0$$

$$\omega^2 = \frac{c}{m} > 0$$

$$f(t) = \frac{F(t)}{m}$$

IC:

$$x(t=0) = \bar{x}_0 \quad \text{initial posn.}$$

$$\left.\frac{dx}{dt}\right|_{t=0} = V_0 \quad \text{initial veloc.}$$

I Soln. of the homog. ODE:

$$\ddot{x} + 2\delta \dot{x} + \omega^2 x = 0$$

Physically this is the case of free oscillations

Char. poly: $x(t) \sim e^{\lambda t}$

$$\lambda^2 + 2\delta\lambda + \omega^2 = 0$$

$$\lambda_{12} = -\delta \pm \sqrt{\delta^2 - \omega^2}$$

Four cases:

① Purely damped motion: $\delta > \omega$

$$x(t) = A e^{\overbrace{\left(-\delta + \sqrt{\delta^2-\omega^2}\right)}^{\text{neg. root}} t} + B e^{\overbrace{\left(-\delta - \sqrt{\delta^2-\omega^2}\right)}^{\text{neg. root}} t}$$

both exponentials have a negative exponent

$\rightarrow x(t) \rightarrow 0$ as $t \rightarrow \infty$ without changing sign.

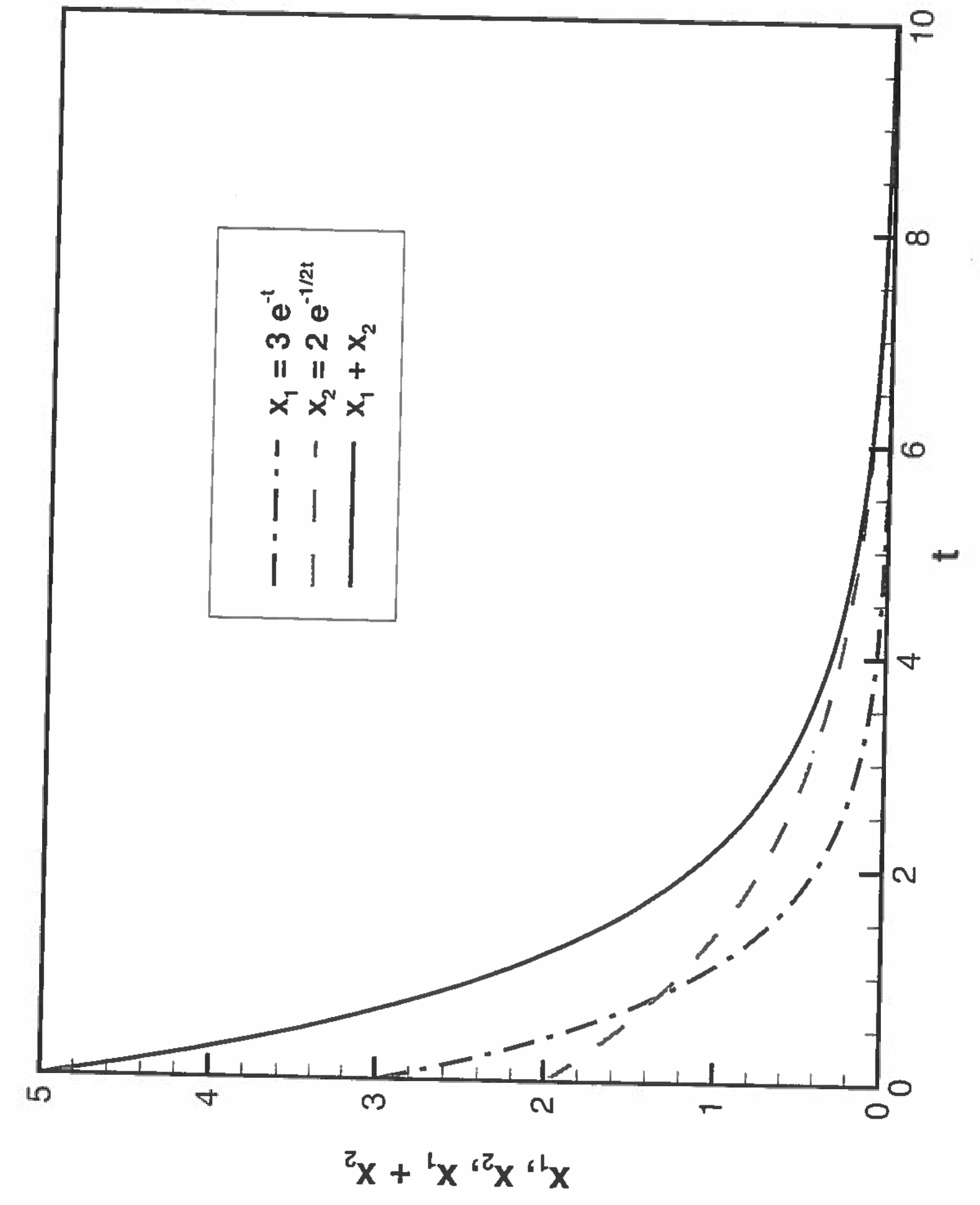


Figure 1: Illustration of a purely damped motion. The mass approaches its equilibrium position $x = 0$ monotonically.

② Critical damping: $\delta = \omega$

repeated root: $\lambda_{12} = -\delta$

$$x(t) = Ae^{-\delta t} + Bte^{-\delta t}$$

$x(t) \to 0$ as $t \to \infty$
with at most one overshoot across $x=0$, depending on I.C.

③ Damped oscillations: $\delta < \omega$

$$\lambda_{12} = -\delta \pm i\sqrt{\omega^2 - \delta^2}$$

$$x(t) = e^{-\delta t}\left(A\cos\left(\sqrt{\omega^2-\delta^2}\,t\right) + B\sin\left(\sqrt{\omega^2-\delta^2}\,t\right)\right)$$

Damped oscillation with frequency $\sqrt{\omega^2-\delta^2}$ & decay rate δ.

oscillations decay like $e^{-\delta t}$; $1/\delta$ is the timescale over which the oscillation decays.

(4) Undamped oscillations: $\delta = 0$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Harmonic oscillation with frequency ω.

ω is the eigenfrequency of the system, i.e. the frequency it oscillates with naturally (without damping)

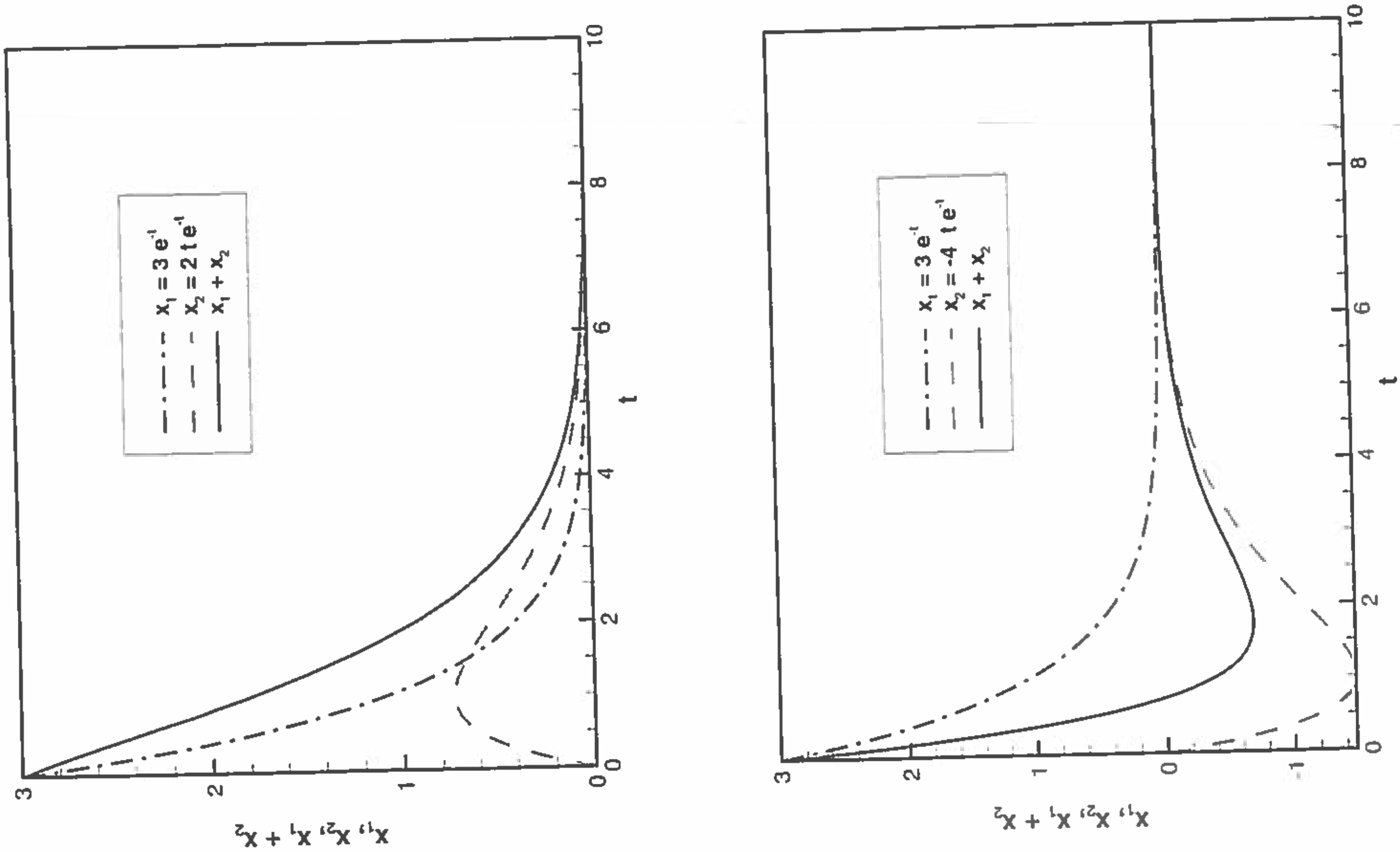


Figure 2: Illustration of critically damped motions. The mass approaches its equilibrium position, $x = 0$, with at most one "overshoot".

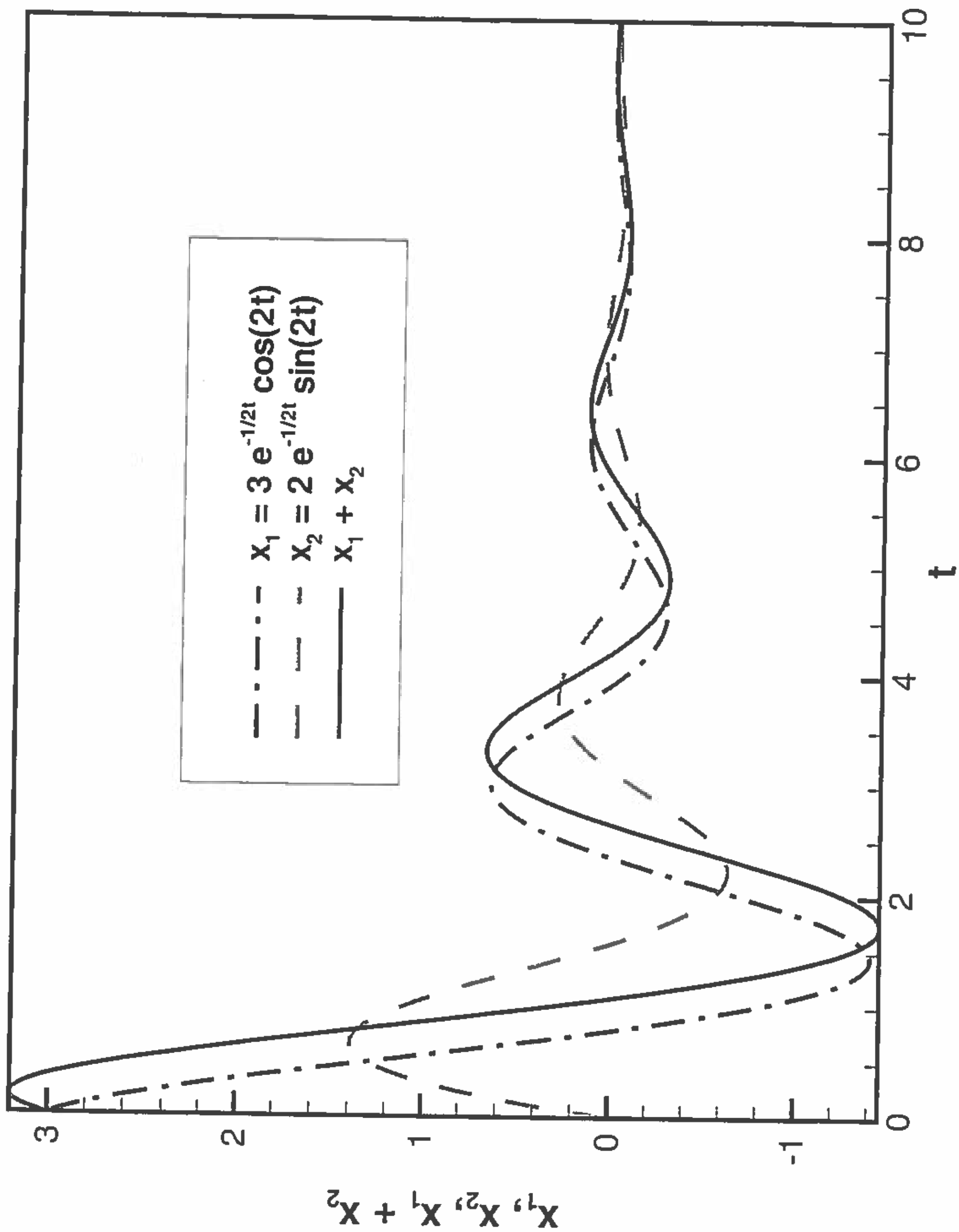


Figure 3: Illustration of a damped oscillation. The mass oscillates about its equilibrium position $x = 0$ and the amplitude of the oscillations decays exponentially.

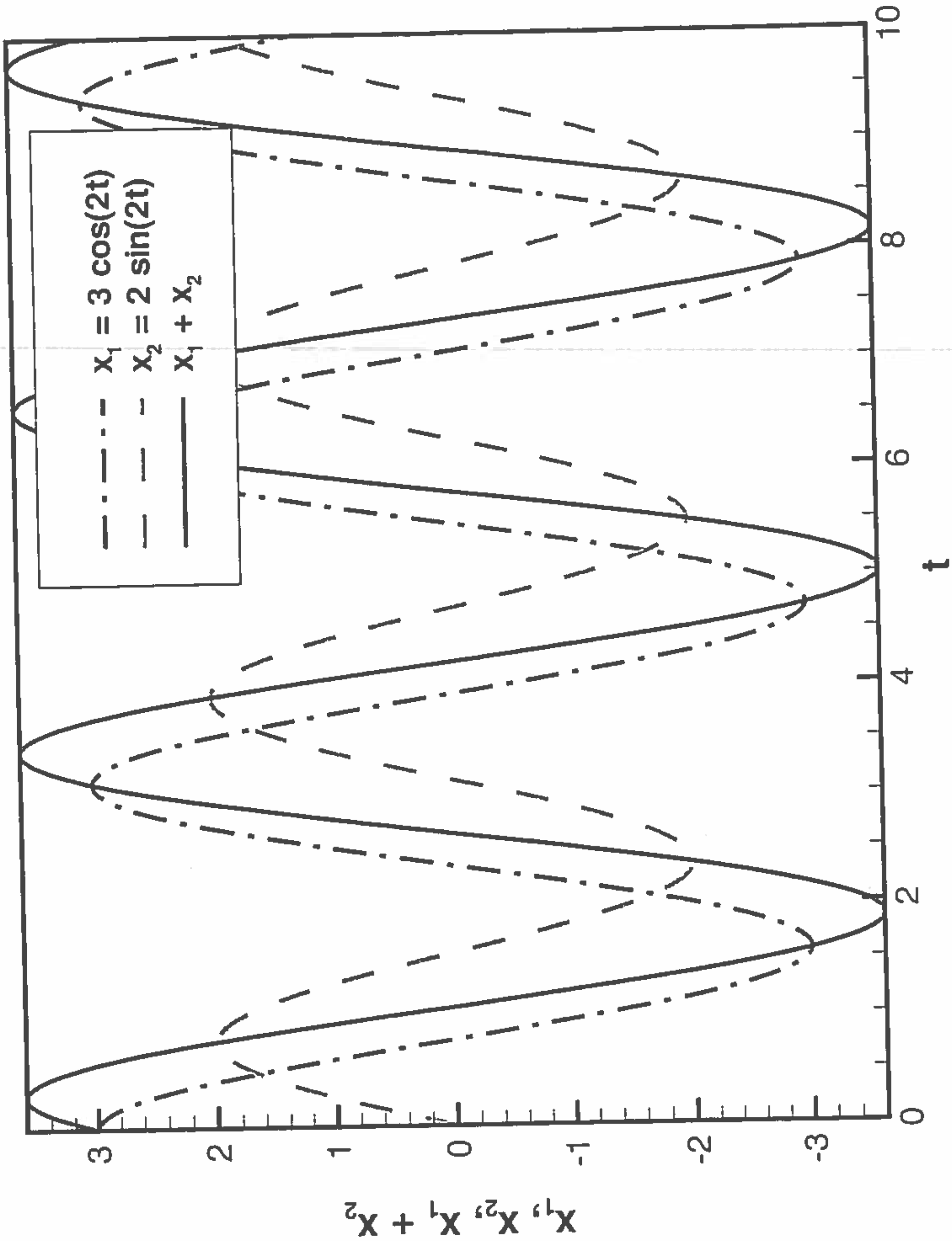


Figure 4: Illustration of an undamped oscillation. The mass performs harmonic oscillations about its equilibrium position $x = 0$.

II Soln. of the inhomog. ODE

Consider periodic forcing
$\Longrightarrow$ resonance.

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = f(t)$$

$$f(t) = \hat{f}\sin(\Omega t)$$

Ω: forcing frequency

or

$$f(t) = \hat{f}\cos(\Omega t)$$

Can do both at the same time by considering exponential forcing

$$f(t) = \hat{f}e^{i\Omega t}$$

Extract real or imag. part of soln. for cos or sin forcing respectively.

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for $f(t) = \hat{f} e^{i\Omega t}$

Ansatz: $x_p = X e^{i\Omega t}$

$$\dot{x}_p = +i\Omega X e^{i\Omega t}$$

$$\ddot{x}_p = -\Omega^2 X e^{i\Omega t}$$

into ODE

$$X \cancel{e^{i\Omega t}} \Big(\underbrace{-\Omega^2}_{\ddot{x}} + 2\delta \underbrace{i\Omega}_{\dot{x}} + \omega^2 \Big) \overset{!}{=} \hat{f} \cancel{e^{i\Omega t}}$$

$$X = \frac{\hat{f}}{(\omega^2 - \Omega^2) + i(2\delta\Omega)}$$ is complex

$$X = X_{real} + i X_{imag}$$

$$= \underline{\underline{|X| e^{i\varphi}}}$$

$$\tan\varphi = \frac{X_{imag}}{X_{real}}$$

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$|\overline{X}|$ represents the amplitude of oscillation:

$$|\overline{X}| = \frac{\hat{f}}{\sqrt{(\omega^2-\Omega^2)^2+(2\delta\Omega)^2}}$$

$$|\overline{X}| = \frac{\hat{f}/\omega^2}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2+\left(2\left(\frac{\delta}{\omega}\right)\left(\frac{\Omega}{\omega}\right)\right)^2}}$$

ratio of forcing frequency to eigenfreq. ω

ratio of damping to spring stiffness or ratio of timescale of free oscillation $1/\omega$ to its decay ~~rate~~ $1/\delta$

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what is $\frac{\hat{f}}{\omega^2} = \frac{\hat{F}\cancel{/m}}{c\cancel{/m}}$

$= \frac{\hat{F}}{c}$ = extension ~~length~~ of the spring when subjected to a steady force $\hat{F}$.