$$y'' + p y' + q y = A_1 r_1(x) + \dots + A_n r_n(x)$$

Want: y_p

Ansatz:

$$y_p = c_1 r_1(x) + \dots + c_n r_n(x)$$

into ODE; collect lin. indep fcts; set coeffns to zero

$\Rightarrow$ n eqns for $c_1 \dots c_n$

Modifications if

- derivs of $r_i(x)$ creates new lin indep fcts
- $r_i(x)$ solves homog. ODE

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$$\ddot{y} + 4y = \underbrace{1}_{A_1} \cdot \underbrace{\cos(3t)}_{r_1(t)} + \underbrace{2}_{A_2} \underbrace{\sin(t)}_{r_2(t)}$$

$$y_p = A\cos(3t) + B\sin(t)$$

$$\vdots$$

$$\cos(3t)(-5A-1) + \sin(t)(3B-2) = 0$$

$$\forall t$$

$$A = -1/5 \qquad B = 2/3$$

Example

$$\ddot{y} + 3\dot{y} + y = \underbrace{4}_{A_1=4} \cdot \underbrace{1}_{r_1(t)=1} + \underbrace{2}_{A_2=2}\underbrace{t^2}_{r_2(t)}$$

Ansatz:

$$y_p = C_1 \cdot 1 + C_2 t^2 + C_3 t$$

$$\dot{y}_p = + 2C_2 t + C_3$$

$$\ddot{y}_p = + 2C_2$$

into ODE:

$$\underbrace{2C_2}_{\ddot{y}} + 3\underbrace{\left(2C_2 t + C_3\right)}_{\dot{y}} +$$

$$+ \underbrace{C_1 + C_2 t^2 + C_3 t}_{y} \overset{!}{=} 4 + 2t^2 \quad \forall t$$

collect lin. indep. fcts:
powers of t:

$$(2c_2 + 3c_3 + c_1 - 4) +$$
$$(6c_2 + c_3)\, t +$$
$$(c_2 - 2)\, t^2 \overset{!}{=} 0 \qquad \forall t$$

Set coeffns to zero

$$c_2 = 2$$
$$c_2 = 0$$

Doesn't work because t' was not in the ansatz for y_p.

Resolve with c_3:

$$c_2 = 2$$
$$c_3 = -12$$
$$c_1 = 36$$

$$y_p = 36 - 12t + 2t^2$$

Example:

$$\ddot{y} + 3\dot{y} = 1 + 9t^2$$

Ansatz:

$$y_p = c_1 + c_2 t + c_3 t^2$$

(see above)

$$\dot{y}_p = c_2 + 2c_3 t$$

$$\ddot{y}_p = 2c_3$$

into ODE

$$\underbrace{2c_3}_{\ddot{y}} + 3\underbrace{\left(c_2 + 2c_3 t\right)}_{\dot{y}} \stackrel{!}{=} 1 + 9t^2$$

collect powers of t:

$$(2c_3 + 3c_2 - 1) + (6c_3)t - 9t^2 \stackrel{!}{=} 0 \quad \forall t$$

doesn't work!

Remedy: multiply by t

$$y_p = c_1 t + c_2 t^2 + c_3 t^3$$

$$\dot{y}_p = c_1 + 2c_2 t + 3c_3 t^2$$

$$\ddot{y} = 2c_2 + 6c_3 t$$

into ODE:

$$\underbrace{2c_2 + 6c_3 t}_{\ddot{y}} + 3\underbrace{\left(c_1 + 2c_2 t + 3c_3 t^2\right)}_{\dot{y}} \overset{!}{=} 1 + 9t^2 \quad \forall t$$

collect powers of t:

$$\left(2c_2 + 3c_1 - 1\right) + \left(\underbrace{6c_3 + 6c_2}_{=0}\right) t + \left(\underbrace{9c_3 - 9}_{=0}\right) t^2 \overset{!}{=} 0 \quad \forall t$$

$c_3 = 1;\ c_2 = -1;\ c_1 = 1$

$$y_p = t - t^2 + t^3 + \text{III.3}$$

(or any other soln. of homog. ODE!)

Sad news: Method only works for some RHS

Counterexample:

$$r(x) = \log(x)$$

Try $y_p = c_1 \log(x) + c_2 x^{-1} + c_3 x^{-2}$

$$\dot{y}_p = c_1 \frac{1}{x} + c_2 (-1 x^{-2})$$

This doesn't work because differentiation of the ansatz keeps generating new lin. indep. fcts

In fact, method only works for

$$r(x) \sim \underbrace{P(x)}_{\text{polynomial}} e^{mx} \begin{Bmatrix} \sin(nx) \\ \cos(nx) \end{Bmatrix}$$

Other methods exist!

- Method of variation of parameters.
- power series expansion

Nonlinear ODEs

2 Special cases:

1 ODE does not depend on y

$$y'' = f(x, \cancel{y}, y') = f(x, y')$$

This is a 1^{st} order ODE for y': Substitution:

$$u(x) = y'(x)$$

ODE becomes:

$$u' = f(x, u)$$

- Solve for $u(x)$ (one const. of integr.)
- then solve

$$\frac{dy}{dx} = u(x) \text{ for } y(x)$$

(second const. of integr.)

Example:

$$3(y')^2 y'' = 1$$

Subst: $u(x) = y'$

$$3u^2 u' = 1$$

$$3u^2 \frac{du}{dx} = 1$$

$$\int 3u^2 \, du = \int dx$$

$$u^3 = x + C$$

$$u = \boxed{(x+C)^{1/3} = \frac{dy}{dx}}$$

$$y(x) = \frac{3}{4}(x+C)^{4/3} + D$$

② Autonomous ODEs

$$y'' = f(\cancel{x}, y, y') = f(y, y')$$

Can be reduced to 1st order ODE for $u(x) = y'$ again if we regard u as a fct of y rather than x

$$y'' = \frac{du}{dx} = \frac{du}{dy}\underbrace{\frac{dy}{dx}}_{u} = u\frac{du}{dy}$$

So ODE becomes

$$u\frac{du}{dy} = f(y, u)$$

is a 1st order ODE for $u(y)$

Then solve

$$\frac{dy}{dx} = u(y(x))$$

<u>Example:</u>

$$y y'' - 2(y')^2 + 2y' = 0$$

is autonomous; ($y = 0$ is a soln)

<u>Subst:</u> $u = y'$

$$y'' = u \frac{du}{dy}$$

into ODE:

$$y u \frac{du}{dy} - 2u^2 + 2u = 0$$

$$u\left(y \frac{du}{dy} - 2u + 2\right) = 0$$

One family of solns is

$$u \equiv 0 = \frac{dy}{dx} \Rightarrow y = \text{const.}$$

Other family:

$$y \frac{du}{dy} = 2(u-1)$$

Seperate:

$$\int \frac{1}{u-1}\, du = \int \frac{2}{y}\, dy$$

$$\ln|u-1| = 2\ln|y| + \hat{C}$$

$$= \ln y^2 + \ln D$$

where $D > 0$

$$= \ln(Dy^2) \quad (*)$$

$$u(y) = \boxed{1 + Dy^2 = \frac{dy}{dx}}$$

Sep:

$$\int dx = \int \frac{1}{1+Dy^2}\, dy$$

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$$x + \hat{c} = \frac{1}{\sqrt{D}} \arctan(\sqrt{D}\, y)$$

$$y(x) = \frac{1}{\sqrt{D}} \tan(\sqrt{D}\, x + c)$$

$$c = \hat{c}\sqrt{D}$$

Postscript:

(*) assumed $u > 1$; $\boxed{u = 1 + Dy^2}$

Alternative: if $u < 1$

$$1 - u = Dy^2$$

$$\boxed{u = 1 - Dy^2}$$

So to cover all these cases:

$$u = 1 + \hat{D} y^2 \quad \text{where } \hat{D} \in \mathbb{R}$$

then the integral gives an arctanh rather than arctan.