$$y'' + \underbrace{\cancel{p(x)}}_{\text{const!}}\, y' + \underbrace{\cancel{q(x)}}_{\text{const;}}\, y = r(x)$$

$$y'' + p\,y' + q\,y = r(x)$$

<u>Gen. soln:</u>

$$y = y_p(x) + \underbrace{A\,y_1(x) + B\,y_2(x)}_{\checkmark}$$

$y_p(x) \downarrow$?

<u>Method of undetermined coeffns</u>

<u>Ex:</u>

$$y'' + p\,y' + q\,y = A\,e^{ax}$$

A, a given

<u>Ansatz:</u> $y = C e^{ax}$

into ODE

$$C = \frac{A}{a^2 + pa + q}$$

But what if

$$a^2+pa+q=0$$

Compare to char. poly:

$$\lambda^2+p\lambda+q=0$$

$\Rightarrow$ problem arises if a is equal to one of the roots of the char. poly.

<u>In that case:</u>

try:

$$y=C\,x\,e^{ax}$$

$$y'=C\,e^{ax}(1+ax)$$

$$y''=C\,e^{ax}\left(a(2+ax)\right)$$

into ODE:

$$\cancel{C e^{ax}}\Big(\underbrace{a(2+ax)}_{y''}+p\underbrace{(1+ax)}_{y'}+\underbrace{q\,x}_{y}\Big)\overset{!}{=}A\,\cancel{e^{ax}}$$

$$C\left(x\underbrace{(a^2+pa+q)}_{=0}+(2a+p)\right) \overset{!}{=} A$$

$$\underline{\underline{C = \frac{A}{2a+p}}}$$

what if $2a+p=0$? $a=-\frac{p}{2}$

This happens if a is a repeated root of the char. poly.

$$\lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

In that case

$$y = Cx^2e^{ax}$$
$$y' = Ce^{ax}(2x+ax^2)$$
$$y'' = Ce^{ax}(2+4ax+a^2x^2)$$

into ODE

$$C\cancel{e^{ax}}\Big((\underbrace{2+4ax+a^2x^2}_{y''}) + p(\underbrace{2x+ax^2}_{y'}) +$$

$$+\,qx^2\Big) \overset{!}{=} A\cancel{e^{ax}}$$

$$C\Big(x^2(\underbrace{a^2+pa+q}_{0}) + x(\underbrace{4a+2p}_{0}) + 2\Big)$$

$$\overset{!}{=} A$$

$$\underline{\underline{C = \frac{A}{2}}}$$

This definitely works.

So, gen. procedure:

$$y'' + py' + qy = Ae^{ax}$$

(1) Solve char. poly:

$$\lambda^2 + p\lambda + q = 0 \Rightarrow \lambda_{12}$$

(2) • If $a \neq \lambda_1$ & $a \neq \lambda_2$

$$y_p = Ce^{ax}$$

• If $a = \lambda_1$ or $a = \lambda_2$ and $\lambda_1 \neq \lambda_2$:

$$y_p = Cxe^{ax}$$

• If $a = \lambda_1 = \lambda_2$

$$y_p = Cx^2e^{ax}$$

<u>Note:</u> Special cases arise if $r(x)$ is a soln. of the homog. ODE!

<u>General approach:</u>

$$y'' + p y' + q y = A_1 r_1(x) + \dots + A_n r_n(x) \quad (*)$$

const. A_i & the lin. indep. fcts $r_i(x)$ are given.

<u>Fact:</u> If $y_1(x)$ is a soln. of

$$y'' + p y' + q y = r_1(x)$$

and $y_2(x)$ is a soln. of

$$y'' + p y' + q y = r_2(x)$$

then $A_1 y_1 + A_2 y_2$ is a soln. of

$$y'' + p y' + q y = A_1 r_1(x) + A_2 r_2(x)$$

(EXERCISE)

undetermined coefficients

<u>Ansatz:</u>

$$y_p = C_1 \underbrace{r_1(x)} + \dots + C_n \underbrace{r_n(x)}$$

given & lin. indep.

<u>Plan:</u> insert into ODE, collect linearly indep. fcts & set their coefficients to zero $\Rightarrow$ n eqns. for $C_1, \dots, C_n$

<u>Modification:</u> If the differentiation of any of the $r_i(x)$ creates new lin. indep. fcts, include these too!

$$\hat{y}_p = C_1 r_1(x) + \dots + C_n r_n(x) + D_1 r_1'(x) + \dots + D_n r_n'(x) + E_1 r_1''(x) + \dots + E_n r_n''(x)$$

(only include new terms)

What can go wrong?

If any of the terms in the ansatz are solns. of the homog. ODE the method won't work!

$$y'' + p y' + q y = A r(x)$$

If
$$r'' + p r' + q r = 0$$
we cannot determine the undetermined coefficient!

$$y_p = C r$$

into ODE

$$C \underbrace{\left(r'' + p r' + q r\right)}_{0} \overset{!}{=} A r$$

In that case try:

$$y_p = c\,x\,r$$
$$y_p' = c\,(r + x r')$$
$$y_p'' = c\,(2r' + x r'')$$

into ODE:

$$c\Big[\underbrace{2r' + x r''}_{y''} + p\underbrace{(r + x r')}_{y'} + q\underbrace{x r}_{y}\Big] \stackrel{!}{=} A r$$

$$c\Big(x\underbrace{(r'' + p r' + q r)}_{\stackrel{!}{=}0} + (2r' + p r)\Big) \stackrel{!}{=} A r$$

This still won't work if
$2r' + pr = 0$

This happens if

$$y_p = C' x r$$

is also a soln. of the homog. ODE!

In that case:

$$y_p = C' x^2 r$$

$$y_p' = C'(2xr + x^2 r')$$

$$y_p'' = C'(2r + 4xr' + x^2 r'')$$

into ODE

$$C'\Big(\underbrace{2r + 4xr' + x^2 r''}_{y''} + \underbrace{2pxr + px^2 r'}_{y'} + \underbrace{q x^2 r}_{y}\Big) \overset{!}{=} Ar$$

$$C\left(x^2\overbrace{\left(r''+pr'+qr\right)}^{0}+x\overbrace{\left(4r'+2pr\right)}^{0}+2r\right)\overset{!}{=}Ar$$

$$\underline{\underline{C=\frac{A}{2}}}$$

So if one of ~~the~~ terms in the ansatz is a soln. of the homog. ODE then multiply that term by x^m where m is the smallest integer for which that product is <u>not</u> a soln. of the homog. ODE.

If this creates yet more lin. indep. fcts add them too.

Example:

$$\underbrace{A_1}\,\underbrace{r_1(t)} + \underbrace{A_2}\,\underbrace{r_2(t)}$$

$$\ddot{y} + 4y = \cos(3t) + 2\sin(t)$$

homog. ODE: $\ddot{y} + 4y = 0$

$$y_H = \hat{A}\cos(2t) + \hat{B}\sin(2t)$$

$$\lambda^2 + 4 = 0;\quad \lambda_{1,2} = \pm 2i$$

Ansatz:

$$y_p = A\,\underbrace{\cos(3t)}_{r_1(t)} + B\,\underbrace{\sin(t)}_{r_2(t)}$$

into ODE:

$$\ddot{y}_p = -9A\cos(3t) - B\sin(t)$$

$$\underbrace{\underline{-9A\cos(3t) - B\sin(t)}}_{\ddot{y}} + \underbrace{\underline{4A\cos(3t)}\quad + 4B\sin(t)}_{y} \overset{!}{=} \underline{\cos(3t)} + \underline{2\sin(t)}$$

collect lin. indep. fcts:

$$\cos(3t)\underbrace{\left(-9A+4A-1\right)}_{\overset{!}{=}0} + \sin(t)\underbrace{\left(-B+4B-2\right)}_{\overset{!}{=}0} = 0 \quad \forall t$$

Because $\sin(t)$ & $\cos(3t)$ are lin. indep. this only possible if the coefficients are zero!

$$\Rightarrow \quad A = -1/5$$
$$B = 2/3$$