\[ y'' + p(x)y' + q(x)y = r(x) \]

**Gen. Soln.**

\[ y = y_p(x) + A y_1(x) + B y_2(x) \]

**Method of Undetermined Coeffs.**

**Ex:**

\[ y'' + p(x)y' + q(x)y = A e^{0x} \]

**Ansatz:**

\[ y = C e^{0x} \]

**into AC**

\[ C = \frac{A}{a^2 + p_a + q} \]
But what if

\[ a^2 + pa + g = 0 \]

compare to char. poly:

\[ x^2 + px + q = 0 \]

A problem arises if \( a \) is equal to one of the roots of the char. poly.

In that case:

Try:

\[ y = C e^{ax} \]

\[ y' = C e^{ax} \cdot (1 + ax) \]

\[ y'' = C e^{ax} \cdot a(2 + ax) \]

Into ODE:

\[
C e^{ax} \left( \frac{a(2 + ax)}{y''} + p \frac{(1 + ax) + q x}{y'} \right)
= A e^{ax}
\]
\[ c \left( x^2 + px + q \right) + (2a + p) = 0 \]
\[ c = \frac{R}{2a+p} \]

What if \( 2a + p = 0 \)?
This happens if \( a = \frac{-b}{2c} \)

\[ \lambda_{12} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - 9} \]

In that case,
\[ y = c x^2 + ax \]
\[ y' = c e^{ax}(2x + ax^2) \]
\[ y'' = c e^{ax}(2 + 4ax + a^2 x^2) \]

\[ \text{in } \theta \text{ of } \eta \]
\[
\begin{align*}
&c \cdot x^2 \left(2 + 4ax + ax^2\right) + p \left(2x + ax^2\right) + \frac{1}{2} dx^2 \\
&= A x^2 \\
&c \cdot (x^2 (a^2 + pa + q) + x^2 (q + 2p + 2)) = A \\
&= 1\sqrt{A} \\
&c = \frac{1\sqrt{A}}{2} \\
&\text{This definitely works.}
\end{align*}
\]
So, perform procedure:

\[ y'' + p y' + q y = Ae^{\alpha x} \]

1. **Solve char. poly:**

\[ \lambda^2 + p\lambda + q = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 \]

2. **If** \( a \neq \lambda_1, \lambda_2 \)

\[ y_p = Ce^{\alpha x} \]

- **If** \( a = \lambda_1, \lambda_2 \) or \( a = \lambda_1 = \lambda_2 \) and \( \lambda_1 \neq \lambda_2 \)

\[ y_p = Ce^{\alpha x} + C' xe^{\alpha x} \]

- **If** \( a = \lambda_1 = \lambda_2 \)

\[ y_p = Cxe^{\alpha x} \]
Note: Special cases arise if \( f(x) \) is a soln. of the homog. ODE!

General approach:

\[
y'' + py' + qy = A_1 f_1(x) + \ldots + A_n f_n(x)
\]

const. \( A_i \) & the lin. indep. fcts \( f_i(x) \) are given.

Fact: If \( f_i(x) \) is a soln. of
\[
y'' + py' + qy = f_i(x)
\]
and \( f_2(x) \) is a soln. of
\[
y'' + py' + qy = f_2(x)
\]
then \( A_1 f_1 + A_2 f_2 \) is a soln. of
\[
y'' + py' + qy = A_1 f_1(x) + A_2 f_2(x)
\]

\[\text{Ex-Example}\]
Ansatz:

\[ y_p = c_1 \Gamma_1(x) + ... + c_n \Gamma_n(x) \]

given & lin. indep.

Plan: insert into ODE, collect linearly indep. fcts & set their coefficients to zero \( \Rightarrow \) n eqns. for \( c_1, \ldots, c_n \)

Modification: If the differentiation of any of the \( \Gamma_i(x) \) creates new lin. indep. fcts, include these too.

\[ y_p = c_1 \Gamma_1(x) + ... + c_n \Gamma_n(x) + \\
D_1 \Gamma'_1(x) + ... + D_n \Gamma'_n(x) + \\
E_1 \Gamma''_1(x) + ... + E_n \Gamma''_n(x) \]

(only include new terms)
What can go wrong?

If any of the terms in the ansatz are non-zero, the homog. ODE the method won't work!

\[ g'' + pg' + qg = \Phi(x) \]

If \( g'' + pg' + qg = 0 \),
we cannot determine the undetermined coefficient!

\[ \Phi = c \Phi \]

in ODE

\[ c \left( g'' + pg' + qg \right) = \Phi \]
In that case try:

\[ \mathbf{t}_\mathbf{P} = C \times \mathbf{r} \]

\[ \mathbf{t}_\mathbf{P} = C (\mathbf{r} + x \mathbf{r}') \]

\[ \mathbf{t}_\mathbf{P} = C (2 \mathbf{r}' + x \mathbf{r}''') \]

into \( \Delta \mathbf{OPE} \):

\[ C \left[ \frac{2 \mathbf{r}'' + x \mathbf{r}'''}{2 \mathbf{r}'} + \mathbf{p} (\mathbf{r} + x \mathbf{r}') + \frac{q \times \mathbf{r}'''}{\mathbf{r}'} \right] \]

\[ = \Delta \mathbf{P} \]

\[ C \left( x (\mathbf{r}'' + p \mathbf{r}') + q \mathbf{r}''') + (2 \mathbf{r}' + p \mathbf{r}') \right) \]

\[ = \Delta \mathbf{P} \]

This still won't work if

\[ 2 \mathbf{r}' + p \mathbf{r} > 0 \]
This happens if
\[ J_p = C \times r \]
is also a soln. of the homog. ODE!

In that case:

\[ J_p = \left( \begin{array}{c} 0 \\ C \end{array} \right) \]
\[ J_p = C \left( \begin{array}{c} 2x + x^2 r \\ 2r + 4x r' + x^2 r'' \end{array} \right) \]

into ODE

\[ C \left( \begin{array}{c} 2r + 4x r' + x^2 r'' \\ 2r + 4x r' + x^2 r'' \end{array} \right) = A r \]
\[ c (x^2 (9 + r^2 + 9r) + x (4r + 2pr) + 2r) = Ar \]

\[ c' = \frac{A}{2} \]

So if one of the terms in the ansatz is a soln. of the homog. ODE then multiply that term by \( x^m \) where \( m \) is the smallest integer for which that product is not a soln. of the homog. ODE.
IF this creates yet
more lin. indep. facts
add them too.

**Example:**

\[ y'' + 4y = \cos(3t) + 2 \sin(t) \]

homog. oDE:

\[ y'' + 4y = 0 \]

\[ y(t) = A \cos(2t) + B \sin(2t) \]

\[ \lambda^2 + 4 = 0; \lambda = \pm 2i \]

**Ansatz:**

\[ y_p = A \cos(2t) + B \sin(2t) \]

\[ \text{into oDE: } r_1(t) \quad r_2(t) \]

\[ y_p'' = -9A \cos(3t) - B \sin(3t) \]
\[-9A \cos(\beta t) - B \sin(\beta t) + 4A \cos(\beta t) = 0\]

\[+ 4B \sin(\beta t) = \cos(\beta t) + 2 \sin(\beta t)\]

Collect even, independent terms:

\[
\begin{align*}
\cos(\beta t) & \left( -9A + 4A - 1 \right) + \\
\sin(\beta t) & \left( -B + 4B - 2 \right) = 0
\end{align*}
\]

At \(t\)

Because \(\sin(\beta t)\) & \(\cos(\beta t)\) are even, independent, this is only possible if the coefficients are zero!

\[A = -\frac{1}{5}\]

\[B = \frac{2}{3}\]