$$y'' + p y' + q y = r(x)$$

p, q: const !

$$y = y_p + \underbrace{A y_1 + B y_2}_{\checkmark}$$

y_p: ?

"Method of undetermined coeffns"

Example:

$$y'' + p y' + q y = A e^{ax}$$

A, a: known

Ansatz: $y_p = C e^{ax}$

C: undet. coeffn.

into ODE:

$$C = \frac{A}{a^2 + p a + q}$$

unless a is root of char. poly.

If so try

$$y = c' x e^{ax}$$
$$y' = c' e^{ax}(1+ax)$$
$$y'' = c' e^{ax}(a(2+ax))$$

into ODE

$$c' \cancel{e^{ax}} \Big(\underbrace{a(2+ax)}_{y''} + p \underbrace{(1+ax)}_{y'} + qx \Big) \overset{!}{=} A \cancel{e^{ax}}$$

$\forall x$

collect x-terms:

$$c' \Big(x \underbrace{(a^2 + pa + q)}_{=0} + (2a+p) \Big) \overset{!}{=} A$$

in this case:

$$\underline{c' = \frac{A}{2a+p}}$$

works unless $2a+p=0$

When does this happen?

Recall: we only used the modified ansatz because a was a root of

$$a^2 + pa + q = 0$$

So

$$a = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

The modified ansatz didn't work because $2a + p = 0$

$$\Rightarrow a = -\frac{p}{2}$$

$\Rightarrow$ So this only happens if a is a repeated root of the char. poly.

In that case try

$$y_p = C x^2 e^{ax}$$

EXERCISE

...

$$C = \frac{1}{2}A$$

& this definitely works!

So general procedure:

$$y'' + py' + qy = Ae^{ax}$$

① Solve homog. ODE via

$$\lambda^2 + p\lambda + q = 0 \Rightarrow \lambda_1, \lambda_2$$

$\Rightarrow$ 2 fundamental solns y_1 & y_2.

② • If $a \neq \lambda_1$ & $a \neq \lambda_2$

$$y_p = Ce^{ax}$$

• If $a = \lambda_1$ or $a = \lambda_2$ & $\lambda_1 \neq \lambda_2$

$$y_p = Cxe^{ax}$$

• If $a = \lambda_1 = \lambda_2$

$$y_p = Cx^2 e^{ax}$$

Note: we have classified the 3 cases in terms of the roots of the char. poly.

Reformulate: Special cases arise if $r(x)$ contains a fct. that is a solution of the homog. ODE.

General approach:

$$y'' + py' + qy = A_1 r_1(x) + A_2 r_2(x) + \dots + A_n r_n(x)$$

where the $r_i(x)$ are lin. indep.

Fact: If $y_1(x)$ is a soln. of

$$y'' + py' + qy = r_1(x)$$

and $y_2(x)$ is a soln. of

$$y'' + py' + qy = r_2(x)$$

then $A_1 y_1 + A_2 y_2$ is a soln. of:

$$y'' + p y' + q y = A_1 r_1(x) + A_2 r_2(x)$$

(EXERCISE)

This suggests the ansatz

undet. coeffns

$$y_p = C_1 r_1(x) + C_2 r_2(x) + \dots + C_n r_n(x)$$

known fcts

<u>Plan:</u> Insert ansatz into ODE, collect lin. indep. fcts & set their coeffns. to zero $\Longrightarrow$ n eqns for $C_1, \dots, C_n$

<u>Modification:</u> If the differentiation of any of the $r_i(x)$ produces new lin. indep. fcts then add them to the ansatz.

$$\hat{y}_P = C_1 r_1(x) + \dots + C_n r_n(x) + D_1 r_1'(x) + \dots + D_n r_n'(x) + E_1 r_1''(x) + \dots + E_n r_n''(x)$$

undef. coeffns

Note: only add new, lin. indep fcts.

What can go wrong?
If any of the terms in the ansatz is a soln. of the homog. ODE this won't work.
To illustrate consider:

$$y'' + py' + qy = A\underbrace{r}_{\text{given}}$$

Ansatz: $y_P = Cr$

into ODE:

$$C(\underbrace{r'' + pr' + qr}_{0}) \stackrel{!}{=} Ar$$

In that case, try

$$y_p = c\,x\,r$$
$$y_p' = c\,(r + x r')$$
$$y_p'' = c\,(2r' + x r'')$$

into ODE:

$$c\Big[\underbrace{2r' + x r''}_{y''} + p\,(\underbrace{r + x r'}_{y'}) + q\,\underbrace{x r}_{y}\Big] \overset{!}{=} A r$$

collect terms involving x

$$c\Big[x\,(\underbrace{r'' + p r' + q r}_{0}) + (2r' + p r)\Big] \overset{!}{=} A r$$

This can work unless $2r' + pr = 0$

This only happens if xr is also a soln. of the homog. ODE!

In that case try

$$y_p = c' x^2 r$$

$$y_p' = c'(2xr + x^2 r')$$

$$y_p'' = c'(2r + 4xr' + x^2 r'')$$

into ODE:

$$c'\Big[\underbrace{2r + 4xr' + x^2 r''}_{y''} + p\underbrace{(2xr + x^2 r')}_{y'} + q x^2 r\Big] \stackrel{!}{=} Ar$$

collect powers of x:

$$c'\Big[x^2 \underbrace{(r'' + pr' + qr)}_{0} + x\underbrace{(4r' + 2pr)}_{0} + 2r\Big] \stackrel{!}{=} Ar$$

$$c' 2r = Ar$$

$$\underbrace{(2c' - A)}_{0} r = 0 \Rightarrow \underline{\underline{c' = \tfrac{1}{2}A}}$$

This always works.

<u>Note:</u> include any new derivatives that appear when differentiating the modified ansatz.

<u>Example:</u> $= A_1 r_1(t) + A_2 r_2(t)$

$A_1 = 1 \quad r_1 = \cos(3t)$

$$\ddot{y} + 4y = \cos(3t) + \underbrace{2}_{A_2}\underbrace{\sin(t)}_{r_2(t)}$$

$$y_H = \hat{A}\sin(2t) + \hat{B}\cos(2t)$$

<u>Ansatz:</u>

$$y_P = A\cos(3t) + B\sin(t)$$

$$\dot{y}_P = -3A\sin(3t) + B\cos(t)$$

$$\ddot{y}_P = -9A\cos(3t) - B\sin(t)$$

into ODE

$$\underbrace{-9A\cos(3t) - B\sin(t)}_{\ddot{y}} + 4\underbrace{(A\cos(3t) + B\sin(t))}_{y} \stackrel{!}{=} \cos(3t) + 2\sin(t)$$

$$\cos(3t)\underbrace{\left(-9A+4A-1\right)}_{\doteq 0}+$$

$$+\sin(t)\underbrace{\left(-B+4B-2\right)}_{\doteq 0}=0$$

$\forall t$

because $\cos(3t)$ & $\sin(t)$ are lin. indep. this requires:

$$-5A=1 \Rightarrow A=-1/5$$

$$3B=2 \Rightarrow B=2/3$$

Gen soln. of the ODE is:

$$y=\underbrace{-\frac{1}{5}\cos(3t)+\frac{2}{3}\sin(t)}_{y_p}+$$

$$+\underbrace{\hat{A}\sin(2t)+\hat{B}\cos(2t)}_{y_H}.$$