The general solution of the inhomogeneous ODE

Theorem

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + A\,y_1(x) + B\,y_2(x),$$

where:

- A and B are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Notes:

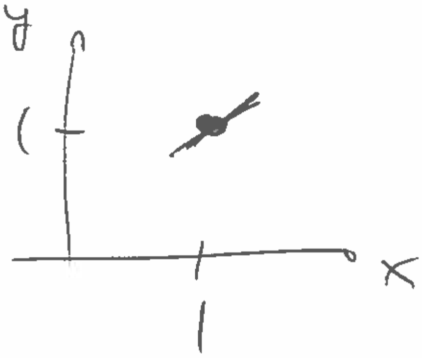
- Note the similarities between the structure of the solution of the linear ODE and the structure of the solution of the linear (algebraic) equation $\mathbf{Ax} = \mathbf{b}$. This is not accidental! There are deep connections between the two fields – matrices and the homogeneous part of a linear ODE are both "linear operators".
- The values of the constants A and B are determined by the boundary or initial conditions.

$$y'' + \underbrace{\frac{1}{x}}_{p(x)} y' \underbrace{- \frac{1}{x^2}}_{q(x)} y = \underbrace{-\frac{1}{x^2}}_{r(x)}$$

<u>IC</u> $y(1) = 1$

$y'(1) = 1$

$x = 1$



<u>EfU:</u>

$$\left.\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \\ r(x) &= -\frac{1}{x^2} \end{aligned}\right\}$$ continuous fcts of x in $I_1 = \mathbb{R}^+$ & $I_2 = \mathbb{R}^-$

$x \in I_1$

$\Longrightarrow$ unique soln exists for $x \in I_1$

<u>Soln:</u> need particular soln

$y_p = 1$ satisfies ODE ✓

<u>Note:</u>

$y_p = 1 + x$ also solves ODE

solns to the homog. ODE (2
(I will call these homogeneous solns.)

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$$

We spot:

$$y_1(x) = x$$

$$y_2(x) = \frac{1}{x}$$

<u>Check:</u> $y_1(x) = x$

$$y' = 1$$

$$y'' = 0 \quad \text{into ODE}$$

$$\underbrace{0}_{y_1''} + \frac{1}{x} \underbrace{1}_{y_1'} - \frac{1}{x^2} \underbrace{x}_{y_1} \stackrel{?}{=} 0 \quad \checkmark$$

$$y_2 = \frac{1}{x} \quad (\text{EXERCISE})$$

y_1 & y_2 are non-zero.

Are they lin. indep.? YES!

Check:

$$A y_1(x) + B y_2(x) = 0 \quad \forall x$$

$$\Longrightarrow A = B = 0 \ ?$$

$$A x + B \frac{1}{x} = 0$$

check at $x=1$: $2A + 2B = 0$

$x=2$: $2A + \frac{1}{2}B = 0$

$$\frac{3}{2}B = 0$$

$$B = 0$$

$$A = 0$$

So y_1 & y_2 are lin. indep.

So the general soln of the ODE is

$$y(x) = \underbrace{1}_{y_p} + A \underbrace{x}_{y_1} + B \underbrace{\frac{1}{x}}_{y_2}$$

A & B follow from ICs

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$$y(1) = 1 + A + B \stackrel{!}{=} 1$$
$$y'(1) = A - B \stackrel{!}{=} 1$$

$$\boxed{y' = A - \frac{B}{x^2}}$$

$$1 + 2A = 2 \Rightarrow A = \tfrac{1}{2}$$

$$B = A - 1 = -\tfrac{1}{2}$$

So soln to IVP is

$$\boxed{y(x) = 1 + \frac{1}{2}x - \frac{1}{2}\frac{1}{x}}$$

<u>Note</u>: Soln is singular at $x=0$

What if we had chosen different fcts y_p, y_1 & y_2:

<u>E.g.</u>:

$y_p = 1 + x$ ← EXERCISE: this solves ODE

$y_1 = x$

$y_2 = x + \frac{2}{x}$

} solve homog. ODE & are nonzero & lin. indep.

Gen. soln:

$$y(x) = \underbrace{1+x}_{y_p} + \hat{A}\,\underbrace{x}_{y_1} + \hat{B}\,\underbrace{\left(x+\frac{2}{x}\right)}_{y_2}$$

$\hat{A}$ & $\hat{B}$ from IC:

$$y' = 1 + \hat{A} + \hat{B}\left(1-\frac{2}{x^2}\right)$$

$$\boxed{\begin{aligned} y(1) &= 2 + \hat{A} + \hat{B}\cdot 3 = 1 \\ y'(1) &= 1 + \hat{A} - \hat{B} = 1 \end{aligned}}$$

$$\Downarrow$$

$$\hat{A} = \hat{B} = -1/4$$

$$y(x) = \underline{1+x} - \frac{1}{4}x - \frac{1}{4}\left(x + \frac{2}{x}\right)$$

$$y(x) = 1 + \frac{1}{2}x - \frac{1}{2}\frac{1}{x}$$

As above!

Note: Soln. is singular at $x = 0$ which is allowed the E & U theorem.

The singularity of the soln. depends on the specific ICs.

Ex: If we choose

$$y(1) = \frac{3}{2}$$

$$y'(1) = \frac{1}{2}$$

$$y(x) = 1 + \frac{1}{2}x \quad \checkmark$$

which is continuous at $x = 0$.