$$y' - xy = x$$

$$y' + p(x)\,y = q(x)$$

Gen. soln:

$$y(x) = \underbrace{-1}_{y_P} + \underbrace{C\exp\left(\tfrac{1}{2}x^2\right)}_{y_H}$$

Observation:

The general soln has two parts:

$$y = y_P + y_H$$

y_P = a (any) particular soln of the full ODE

y_H = general soln of the homogeneous ODE, i.e. the ODE for $q(x) = 0$.

In our example:

- $y_P = -1$ into ODE: $y' - xy = x$

  $y_P' = 0$

  $$(-x)(-1) \stackrel{?}{=} x \quad \checkmark$$

- y_H :

$$y_H' - x\,y_H \stackrel{?}{=} 0$$

$$\frac{dy_H}{dx} = x\,y_H$$

$$\int \frac{1}{y_H}\,dy_H = \int x\,dx$$

$$\ln|y_H| = \frac{1}{2}x^2 + A$$

$$y_H = \exp\left(A + \frac{1}{2}x^2\right)$$

$$= \underbrace{\exp(A)}_{C} \exp\left(\frac{1}{2}x^2\right)$$

$$\underline{y_H = C \exp\left(\frac{1}{2}x^2\right)}$$

This structure is general for all linear ODES.

Some theory for *linear* 2nd order ODEs

Existence and Uniqueness

Consider the *linear* second-order ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x), \qquad (1)$$

subject to the initial conditions

$$y(X) = Y, \quad y'(X) = Z, \qquad (2)$$

where the constants X, Y and Z, and the functions $p(x)$, $q(x)$ and $r(x)$ are given.

Theorem

> If the functions $p(x)$, $q(x)$ and $r(x)$ are continuous functions of x in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in the entire interval I.

Notes:

- This is the promised extension of the statement for first-order problems. The extension to even higher-order linear ODEs should be obvious...
- If the functions $p(x)$, $q(x)$ and $r(x)$ are "well-behaved" (no jumps, singularities, etc.), the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

The homogeneous ODE & superposition of its solutions

If we set $r(x) = 0$ in the *inhomogeneous* ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x), \qquad \text{(I)}$$

we obtain the corresponding *homogeneous* ODE

$$y'' + p(x)\,y' + q(x)\,y = 0. \qquad \text{(H)}$$

A trivial (?) but useful observation

If $y_1(x)$ and $y_2(x)$ are two solutions of (H) then the linear combination

$$y_3(x) = A\,y_1(x) + B\,y_2(x)$$

is also a solution, for any values of the constants A and B.

Linear independence

To see why this is a useful observation, we need to define the concept of linear independence: Two nonzero functions $y_1(x)$ and $y_2(x)$ are linearly independent if

$$A\,y_1(x) + B\,y_2(x) = 0 \quad \forall x \qquad \Longleftrightarrow \qquad A \equiv B \equiv 0$$

(...just as in linear algebra...).

Examples:

- $y_1(x) = x$ and $y_2(x) = 3\,x^2$ are linearly independent.
- $y_1(x) = x$ and $y_2(x) = 3\,x$ are linearly dependent – they're just multiples of each other.

$$y'' + p y' + q y = 0$$

y_1 & y_2 are solns.

<u>Claim:</u> $y = A y_1 + B y_2$ is also a soln.

<u>Proof:</u> Into ODE:

$$y' = A y_1' + B y_2'$$

$$y'' = A y_1'' + B y_2''$$

$$\underbrace{\underline{A y_1''} + \underline{B y_2''}}_{y''} + p\underbrace{\left(\underline{A y_1'} + \underline{B y_2'}\right)}_{y'} + q\underbrace{\left(\underline{A y_1} + \underline{B y_2}\right)}_{y} \overset{?}{=} 0$$

collect A & B:

$$A\left(\cancel{y_1'' + p y_1' + q y_1}\right) + B\left(\cancel{y_2'' + p y_2' + q y_2}\right) \overset{?}{=} 0$$

✓

$$y_1 = x \, ; \quad y_2 = 3x^2$$

$$A\underbrace{x}_{y_1} + B\underbrace{3x^2}_{y_2} = 0 \qquad \forall x$$

Check for some random points:

$\underline{x=1:}$ $\quad A + 3B = 0$

$\underline{x=-1:}$ $\quad -A + 3B = 0$

$$6B = 0 \Rightarrow B = 0$$

$$A = 0$$

$\Rightarrow$ y_1 & y_2 are lin. indep.

$$y_1 = x \, ; \quad y_2 = 3x$$

$$A y_1 + B y_2 = 0$$

$$Ax + B\,3x = 0$$

$$x(A+3B) = 0 \qquad \forall x$$

e.g. $B = 1, \; A = -3$

$\Rightarrow y_1$ & y_2 are lin. dep. !

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Fundamental solutions of the homogeneous ODE

Theorem

Any solution of the homogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = 0. \qquad \text{(H)}$$

can be written as a linear combination of *any* two non-zero, linearly independent solutions, $y_1(x)$ and $y_2(x)$, say:

$$y(x) = A\,y_1(x) + B\,y_2(x).$$

The two non-zero, linearly independent solutions $\{y_1(x),\, y_2(x)\}$ are called "fundamental solutions" of the homogeneous ODE (H).

Notes:

- The set of fundamental solutions is not unique!

The general solution of the inhomogeneous ODE

Theorem

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + A\, y_1(x) + B\, y_2(x),$$

where:

- A and B are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Notes:

- Note the similarities between the structure of the solution of the linear ODE and the structure of the solution of the linear (algebraic) equation $\mathbf{Ax} = \mathbf{b}$. This is not accidental! There are deep connections between the two fields – matrices and the homogeneous part of a linear ODE are both "linear operators".
- The values of the constants A and B are determined by the boundary or initial conditions.

Example:

$$y'' + \underbrace{\frac{1}{x}}_{p} y' - \underbrace{\frac{1}{x^2}}_{q} y = -\underbrace{\frac{1}{x^2}}_{r}$$

IC: $y(1) = 1$

$y'(1) = 1 \qquad x > 1$

$$\left.\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \\ r(x) &= -\frac{1}{x^2} \end{aligned}\right\}$$

continuous fcts of x in $I = \mathbb{R}^+$

$\Rightarrow$ unique soln exists for $x \in \mathbb{R}^+$

- particular soln:

$y_p = 1$