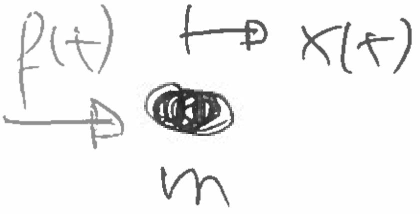


$$m \frac{d^2 x}{dt^2} = f(t)$$

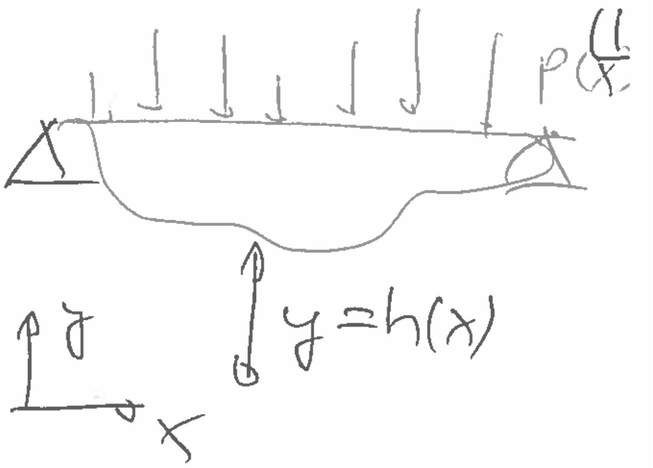
IC: $x(t=0) = x_0$

$$\left.\frac{dx}{dt}\right|_{t=0} = v_0$$

IVP

$$x(t) = \frac{1}{2}\frac{F_0}{m} t^2 + v_0 t + x_0$$

for $f(t) = F_0 = \text{const}$



$$T \frac{d^2 h}{dx^2} = \rho(x)$$

BC: $h(x=0) = 0$

$h(x=L) = L$

BVP

$$h(x) = \frac{1}{2}\frac{\rho_0}{T}(x^2 - Lx)$$

for $\rho(x) = \rho_0 = \text{const}$

E & U: "obvious" (physically)

Finally a counterexample for uniqueness:

$$\boxed{y' = y^{1/2} \qquad y(0) = 0}$$

one IC for 1^{st} order ODE

$y \equiv 0$ is a soln

$$y = \frac{1}{4} x^2$$

check: $y^{1/2} = \frac{1}{2} x$

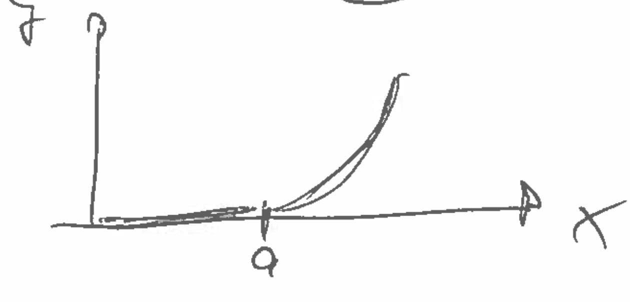
ODE

$y' = \frac{1}{2} x$

IC

In fact:

$$y(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \frac{1}{4}(x-a)^2 & \text{for } x > a \end{cases}$$



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Existence and uniqueness theorem for 1st order ODEs

Consider the first-order ODE in its explicit form

$$\frac{dy}{dx} = f(x, y), \qquad (1)$$

subject to the initial condition

$$y(X) = Y, \qquad (2)$$

where the constants X and Y are given.

Theorem

If $f(x, y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous functions of x and y in a region $0 < |x - X| < a$ and $0 < |y - Y| < b$, then there **exists exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

Notes:

- The statement is easily generalised to higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

A pretty weak statement then....

Example:

$$y' = \underbrace{\sin(xy)}_{f(x,y)} \qquad y(0) = 1$$

$\bar{x} = 0 \qquad \bar{y} = 1$

$$f(x,y) = \sin(xy)$$
$$\frac{\partial f}{\partial y} = x\cos(xy)$$

both continuous fcts of x & y everywhere, especially near $\bar{x}, \bar{y}$.

$\Longrightarrow$ unique soln exists in the vicinity of $x=0$. !

Even though the exact solution is not known!

Ex: $y' = \underbrace{y^2}_{f(x,y)} \qquad y(0) = 1$

$\bar{x} = 0 \qquad \bar{y} = 1$

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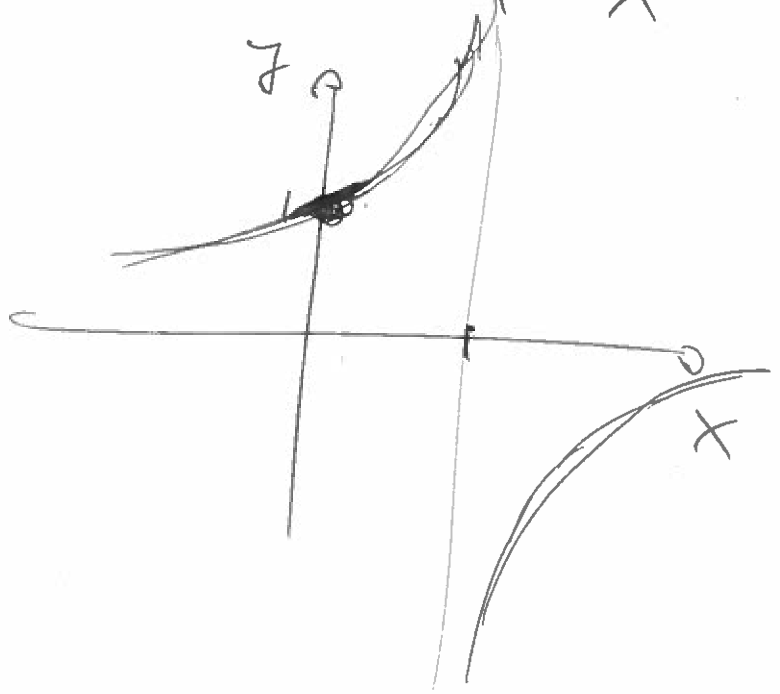
$$f(x,y) = y^2$$
$$\frac{\partial f}{\partial y} = 2y$$
} both continuous fcts of x & y for all x & y.

$\Rightarrow$ unique soln. exists near $x = 0$.

<u>Note:</u>

$y = \frac{1}{1-x}$ is the soln!



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Existence and uniqueness theorem for *linear* 1st order ODEs

Consider the *linear* first-order ODE

$$\frac{dy}{dx} + p(x)\, y = q(x), \tag{3}$$

subject to the initial condition

$$y(X) = Y, \tag{4}$$

where the constants X and Y and the functions $p(x)$ and $q(x)$ are given.

Theorem

> If the functions $p(x)$ and $q(x)$ are continuous functions in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (3) and (4) in the entire interval I.

Notes:

- The statement is again easily generalised to higher-order ODEs.
- The theorem provides a "much more global" statement. In fact, if the functions $p(x)$ and $q(x)$ are "well-behaved" (no jumps, singularities, etc.) the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

This is a much stronger statement and explains in part why (some) mathematicians love (only) linear problems.

Examples:

$$\underbrace{y'+xy}_{p(x)} = \underbrace{x}_{q(x)} \qquad y(0)=2$$

$x=0$

$y=2$

$$\left.\begin{aligned} p(x) &= x \\ q(x) &= x \end{aligned}\right\}$$ both continuous fcts of x for $x \in \mathbb{R}$.

$\Rightarrow$ unique soln exists for $x \in \mathbb{R}$.

In fact: $y(x) = 1 + \exp\left(-\frac{1}{2}x^2\right)$ is the soln.

Example:

$$y' + \underbrace{\frac{1}{x}}_{p(x)} y = \underbrace{2}_{q(x)}$$

$$\left.\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 2 \end{aligned}\right\}$$ are continuous fcts of x in two intervals:

$I_1 = (-\infty, 0)$ & $I_2 = (0, \infty)$

If $x \in I_1 \Rightarrow$ soln exists
& is unique
in I_1

$x \in I_2 \Rightarrow \ldots I_2$

In fact: The general soln of
ODE is given by

$$u(x) = x + \frac{A}{x}$$

A = const follows from I.C.

If $A \neq 0$: singularity at $x = 0$
$A = 0$: soln. exists for
$x \in \mathbb{R}$.