MATHT10222: EXAMPLE SHEET[1] VI

Questions for supervision classes

Hand in the solutions to questions 1(a,b), 2(a) and 3. Attempt all other questions too and raise any problems with your supervisor.

1. **An algebraic example for perturbation methods: Roots of polynomials**

In the lecture we motivated the use of perturbation methods by considering the roots of the quadratic polynomial $x^2 + \epsilon x - 1 = 0$ for small values of ϵ. Examination of the Taylor expansion of the (known) exact solution $x_{[1,2]} = -\epsilon/2 \pm \sqrt{(\epsilon/2)^2 + 1}$ suggested looking for a solution in the form of a power series in ϵ,

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ..., \tag{1}$$

where the unknown coefficients $x_0, x_1, x_2, ...$ were determined via a sequence of simple algebraic equations. The result obtained by this method was shown to be very accurate for small ϵ but you may ask "What's the point of this? We already knew the exact solution!".

To answer this question, we shall now apply the method to obtain an approximation for the roots of the fourth-order polynomial

$$x^4 + \epsilon x - 1 = 0 \tag{2}$$

for $|\epsilon| \ll 1$. This is much more interesting because it is fiendishly difficult (though not *completely* impossible) to obtain a closed-form solution to this problem. [Note that Galois showed that for fifth- (and higher-)order polynomials, it *is* impossible to find closed-form solutions for the roots, unless the polynomial's coefficients have some special structure. Perturbation (or numerical) methods are your best friends for such problems!]

(a) Determine the first three terms, x_0, x_1 and x_2, in the expansion (1) for the polynomial (2), using the same methodology that we employed in the lecture. [**Note:** The leading-order equation that determines the value of x_0 has the same four roots ($x_{[1,2,3,4]} = \pm 1, \pm i$) as the polynomial (2) for $\epsilon = 0$; restrict yourself to the case $x_0 = 1$ when determining the higher-order corrections x_1 and x_2.]

(b) Show that for $\epsilon = 0.2$ the three-term approximation $x = 1 - 1/4\,\epsilon - 1/32\,\epsilon^2$, obtained in part 1a, agrees with the exact value 0.94876 (obtained numerically) to 4 significant figures.

(c) To show that straightforward (so-called regular) perturbation expansions like (1) don't always work, analyse the behaviour of the roots

$$x_{[1,2]} = -\frac{1}{2\epsilon}\left(1 \pm \sqrt{1 + 4\epsilon}\right)$$

[1] Any feedback to: *M.Heil@maths.manchester.ac.uk*

of the quadratic polynomial

$$\epsilon x^2 + x - 1 = 0$$

in the limit of $\epsilon \to 0$. Can the roots be approximated by an expansion of the form $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$? Can you explain what goes wrong?

2. **Perturbation methods for linear ODEs: A mechanical oscillator with a weak spring**

   (a) Use the perturbation expansion

   $$x(t) = x_0(t) + \epsilon\, x_1(t) + \epsilon^2\, x_2(t) + ...,$$

   to determine an approximate solution of the initial value problem comprising the linear ODE

   $$\ddot{x} + \dot{x} + \epsilon x = 0,$$

   for $|\epsilon| \ll 1$, subject to the initial conditions

   $$x(t=0) = 1 \quad \text{and} \quad \dot{x}(t=0) = 0.$$

   This IVP is a model of a mechanical oscillator with a weak spring. Calculate the first three terms in the expansion, i.e. $x_0(t), x_1(t)$ and $x_2(t)$.

   (b) Use your favourite plotting package to assess the quality of the approximate solutions in the range $t \in [0, 10]$ for $\epsilon = 0.2$.

3. **Perturbation methods for non-linear ODEs: Getting rid of dead cats**

The stockbroker referred to in Q2 on the previous example sheet has realised that observing the bounce (or non-bounce) of dead cats does not allow an accurate prediction of the share price. (Sadly, most other methods don't do much better...). He therefore decides to get rid of his dead cat by catapulting it vertically into the air, hoping that, if flung hard enough (stockbrokers are rich and can afford very strong catapults!), the cat might escape the earth's gravitational field and disappear into outer space.

In the second part of the course Rich Hewitt may (or may not[2]) demonstrate that that the cat's motion is described by an initial value problem of the form

$$\ddot{x} + \frac{1}{(1 + \epsilon x)^2} = 0,$$

subject to the initial conditions

$$x(t=0) = 0 \quad \text{and} \quad \dot{x}(t=0) = 1,$$

[2]If he doesn't, and if you're keen to find out more: The derivation is in Lin & Segel's excellent book "Mathematics Applied to Deterministic Problems in the Natural Sciences" – probably the best maths book I own! Rather boringly, Lin & Segel deal with projectiles rather than dead cats but the maths is the same...

where $x(t)$ is a measure of the cat's height above the earth's surface.

Use the perturbation expansion

$$x(t) = x_0(t) + \epsilon\, x_1(t) + \epsilon^2\, x_2(t) + ...,$$

to determine an approximate solution for the cat's trajectory for $|\epsilon| \ll 1$.

Hint: Use the binomial series

$$(1 + \epsilon x)^{-2} = 1 - 2(\epsilon x) + 3(\epsilon x)^2 - ...$$

(valid for $|\epsilon x| < 1$) to re-write the ODE before inserting your perturbation expansion, otherwise the algebra gets horrendous.