MATH10222: EXAMPLE SHEET[1] III

Questions for supervision classes

Hand in the solutions to questions 1, 2a,b and 3a-f. [Feel free to skip the application of the initial conditions in question 3 once you're sure you know how it works. By the end of question 3 you should (i) be pretty bored with homogeneous constant-coefficient ODEs, and (ii) be able to do them in your sleep!] Attempt all other questions too. Questions 2c and 4 should be pretty straightforward; question 5 provides a more detailed (and interesting!) analysis of the "repeated roots" case.
As always, raise any problems with your supervisor.

1. **Existence and uniqueness for linear second-order ODEs**

(a) Does the initial value problem

$$x^2\, y'' - 2\, x\, y' + 2\, y = 0,$$

subject to

$$y(x=1) = 1 \quad \text{and} \quad y'(x=1) = 2$$

have a unique solution? If so, specify the interval in which the solution is guaranteed to exist.

(b) Does the initial value problem

$$\ddot{x} - 2\,\frac{1}{t}\,\dot{x} + 2\,\frac{1}{t^2}\,x = 0,$$

subject to

$$x(t=-1) = 1 \quad \text{and} \quad \dot{x}(t=-1) = 2$$

have a unique solution? If so, specify the interval in which the solution is guaranteed to exist.

(c) Does the initial value problem

$$\ddot{y} + \Omega^2\, y = 0,$$

subject to

$$y(t=0) = 0 \quad \text{and} \quad \dot{y}(t=0) = 0,$$

where Ω is a given constant, have a unique solution? If so, specify the interval in which the solution is guaranteed to exist. Can you spot the solution without doing any calculations?

[1] Any feedback to: *M.Heil@maths.manchester.ac.uk*

(d) Now consider the same ODE as in the previous example, but in the context of the *boundary* value problem

$$\ddot{y} + \Omega^2 \, y = 0,$$

subject to

$$y(t=0) = 0 \quad \text{and} \quad y(t=1) = 0$$

where Ω is a given constant. Recall that the existence and uniqueness theorem does not apply to boundary value problems. Show that the solution of the initial value problem of question 1c is also a solution of the above boundary value problem, demonstrating the existence of *a* solution. Is it possible that there are other solutions? [Hint: Consider the special cases $\Omega = \pi, 2\pi, ...$]

2. **Linear and nonlinear second-order ODEs**

(a) Verify that $y_1(t) = t$ and $y_2(t) = t^2$ are linearly independent solutions of the ODE

$$t^2 \, \ddot{y} - 2 \, t \, \dot{y} + 2 \, y = 0,$$

then write down its general solution.

(b) Verify that $y_1(t) = e^t$ and $y_2(t) = e^{2\,t}$ are linearly independent solutions of the ODE

$$y \, \ddot{y} - (\dot{y})^2 = 0,$$

but that $y = A \, y_1(t) + B \, y_2(t)$, where A and B are arbitrary constants, is not a solution. Explain why.

(c) Prove the statement that if $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous linear ODE

$$y'' + p(x) \, y' + q(x) \, y = 0,$$

then the linear combination $A \, y_1(x) + B \, y_2(x)$ is also a solution, for *any* values of the constants A and B.

3. **Homogeneous linear ODEs with constant coefficients**

Solve the following initial value problems:

(a) $\ddot{y} - 5 \, \dot{y} + 4 \, y = 0$ subject to $y(0) = 0, \quad \dot{y}(0) = 1.$

(b) $\ddot{y} + 4 \, y = 0$ subject to $y(0) = 1, \quad \dot{y}(0) = 0.$

(c) $\ddot{y} - y = 0$ subject to $y(0) = 1, \quad \dot{y}(0) = 0.$

(d) $\ddot{y} + 4 \, \dot{y} + 4 \, y = 0$ subject to $y(0) = 1, \quad \dot{y}(0) = -2.$

(e) $\ddot{y} - 2 \, \dot{y} + 3 \, y = 0$ subject to $y(0) = 0, \quad \dot{y}(0) = \sqrt{2}.$

(f) $\ddot{y} = 0$ subject to $y(0) = 1, \quad \dot{y}(0) = -2.$

4. **The real form of the fundamental solutions in the case of complex conjugate roots of the characteristic polynomial**

   Consider the fundamental solutions of the homogeneous ODE

   $$y'' + p\, y' + q\, y = 0,$$

   where the constants p and q are such that $q > (p/2)^2$. In the lecture we showed that the general solution of the ODE was given by

   $$y(x) = e^{\mu x}\left(\widehat{A}\, e^{i\omega x} + \widehat{B}\, e^{-i\omega x}\right), \tag{1}$$

   where $\omega = \sqrt{q - (p/2)^2}$ and $\mu = -p/2$. If we are only interested in real solutions, the constants $\widehat{A}$ and $\widehat{B}$ obviously have to be complex. In the lecture we had argued (rather convincingly, but indirectly) that it must possible to re-write the real solution in the form

   $$y(x) = e^{\mu x}\left(A\, \cos(\omega x) + B\, \sin(\omega x)\right).$$

   where $A, B \in \mathbb{R}$.

   Prove this by "brute force" calculation. [**Hint:** Write $\widehat{A}$ and $\widehat{B}$ in terms of their real and imaginary parts, $\widehat{A} = \alpha + i\beta$ and $\widehat{B} = \gamma + i\delta$, say, where the constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Insert into (1) and expand, then set the imaginary part of the resulting expression to zero.]

5. **The form of the solution for repeated roots – "reduction of order"**
   The characteristic polynomial for the homogeneous linear ODE

   $$\ddot{y} + 2\,k\,\dot{y} + k^2\,y = 0, \tag{2}$$

   has a repeated root $\lambda = -k$. One of the two fundamental solutions is therefore given by $y_1(t) = e^{-kt}$. We demonstrated in the lecture (by "brute force" checking) that $y_2(t) = t\, e^{-kt}$ is a second, linearly independent solution. What motivated your lecturer to suggest this as a possible solution?[2]

   To solve this mystery, we will now demonstrate a systematic way of constructing a second solution, $y_2(t)$, to a homogeneous, second-order linear ODE if one solution, $y_1(t)$, is already known. The method (known as the "reduction of order") is to look for a solution of the form $y_2(t) = g(t)\, y_1(t)$, where $g(t)$ is an unknown function. Inserting this ansatz into the second-order ODE produces a first-order linear ODE for $\dot{g}(t)$ that can be integrated with standard methods (e.g. using the integrating factor method).

   Try this method for the ODE (2) and thus show that its general solution may indeed be written as $y(t) = (C + D\,t)\, e^{-k\,t}$, where C and D are constants.

[2] Well, your lecturer is obviously very very clever, but do you really think he's clever enough to simply have spotted this?