

3 Second-Order Ordinary Differential Equations

The general form of a second-order ODE is given by

$$\mathcal{F}(x, y(x), y'(x), y''(x)) = 0.$$

It is typically augmented by two boundary or initial conditions, i.e constraints of the form

$$y(X) = Y, \quad y'(X) = Z,$$

or

$$y(X_1) = Y_1, \quad y(X_2) = Y_2$$

where the constants X, Y, Z (or X_1, Y_1, X_2, Y_2) are given.

Often the ODE can be written in explicit form as

$$y''(x) = f(x, y(x), y'(x)).$$

In this lecture we will mainly concentrate on linear second-order ODEs. (In section 3.3 we will briefly discuss the solution of two particular types of nonlinear ODEs).

3.1 Some theory for linear second-order ODEs

- In general, we shall write a *linear* second-order ODE for $y(x)$ in one of two ways, either as

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

or as

$$y'' + p(x)y' + q(x)y = r(x).$$

We will take these ODEs to be defined on an interval

$$I = (\alpha, \beta) = \{x \mid \alpha < x < \beta\}$$

which is chosen such that, at all values of x in I :

- $a(x), b(x), c(x)$ and $d(x)$ are defined and continuous
- and $a(x)$ is never zero

so that the functions $p(x), q(x)$ and $r(x)$, which are defined as

$$p(x) = b(x)/a(x), \quad q(x) = c(x)/a(x), \quad r(x) = d(x)/a(x),$$

are also defined and continuous throughout I .

- **Theorem (*Existence and Uniqueness*):** If $y(x)$ satisfies the ODE $y'' + p(x)y' + q(x)y = r(x)$, and the functions $p(x), q(x)$ and $r(x)$ are continuous throughout the interval I , then there is only one solution that satisfies the pair of initial conditions

$$y(X) = Y \quad \text{and} \quad y'(X) = Z$$

and this solution exists throughout the interval I .

This theorem guarantees that solutions will exist throughout the interval I and that the two initial conditions, one giving the value of y and the other giving the value of its derivative y' , both specified at the same point in I , are enough to select a unique solution.

Note that the existence and uniqueness theorem only applies to initial value problems!

- **Superposition:** In the special case in which the ODE has $r(x)$ set equal to zero, that is for the special form of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

which is known as the ‘homogeneous’ form of the ODE, a linear combination of any solutions is also a solution. Thus if $y_1(x)$ and $y_2(x)$ are solutions of $y'' + p(x)y' + q(x)y = 0$ then so is any function that can be written as $Ay_1(x) + By_2(x)$ for any constants A and B .

- **Fundamental Solutions:** What is more, *any solution* of the homogeneous second-order linear ODE $y'' + p(x)y' + q(x)y = 0$ can be written as a linear combination of only two solutions $y_1(x)$ and $y_2(x)$, known as ‘fundamental solutions,’ provided $y_1(x)$ and $y_2(x)$ are nonzero and linearly independent.

[Reminder: Two functions $y_1(x)$ and $y_2(x)$, defined on I , are said to be **linearly independent** on I if the only linear combination of them that adds up to zero, so that $Ay_1(x) + By_2(x) = 0$ for all $x \in I$, is the one for which $A = B = 0$.]

The choice of fundamental solutions is not unique. For instance, if $\{y_1(x), y_2(x)\}$ is a set of fundamental solutions for a given linear homogeneous ODE then $\{y_1(x), (y_1(x) + y_2(x))\}$ is another set of fundamental solutions.

A solution of the homogeneous ODE is sometimes called a *complementary function*.

- **General Solutions:** Any solution of the non-homogeneous ODE $y'' + p(x)y' + q(x)y = r(x)$ has the form, known as the ‘general solution’

$$y = y_P(x) + Ay_1(x) + By_2(x)$$

where $y_P(x)$, known as a ‘particular solution,’ is a solution of the non-homogeneous ODE, and $y_1(x)$ and $y_2(x)$ are fundamental solutions of the homogeneous form of the ODE, in which $r(x)$ is set to zero.

- The solution to a specific boundary or initial value problem can therefore be obtained in four steps:
 1. Find the general solutions of the homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0 \implies y_H(x) = Ay_1(x) + By_2(x),$$

where $y_1(x)$ and $y_2(x)$ are two nonzero, linearly independent solutions, i.e. they are fundamental solutions of the homogeneous ODE.

2. Find a particular solution of the inhomogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x) \implies y_P(x).$$

3. Write down the general solution

$$y(x) = y_P(x) + y_H(x) = y_P(x) + Ay_1(x) + By_2(x).$$

4. Determine the constants A and B from the boundary or initial conditions.

3.2 Linear second-order ODEs with constant coefficients

3.2.1 The general solution of the homogeneous ODE

- Second-order ODEs for $y(x)$ of the form

$$y'' + py' + qy = 0 \quad \text{with } p \text{ and } q \text{ constant}$$

can always be solved, for all real values of x , using the ansatz

$$y = e^{\lambda x}.$$

[**Important:** The method does not generally work when p and q are not constant.]

- Inserting $y = e^{\lambda x}$ into the ODE and cancelling the common factor $e^{\lambda x}$ yields the so-called *characteristic polynomial*

$$\lambda^2 + p\lambda + q = 0 \quad \text{with roots } \lambda = \frac{1}{2}(-p \pm \sqrt{p^2 - 4q}).$$

The roots, and hence the nature of the solutions, depends on the sign of the ‘discriminant’ $p^2 - 4q$:

Case 1: $p^2 - 4q > 0$

If the discriminant is positive ($p^2 - 4q > 0$) then λ has two distinct real roots of the form

$$\lambda_1 = \frac{1}{2}(-p - \sqrt{p^2 - 4q}) \quad \text{and} \quad \lambda_2 = \frac{1}{2}(-p + \sqrt{p^2 - 4q}).$$

The general solution of the homogenous ODE can therefore be written as

$$y = A e^{\lambda_1 x} + B e^{\lambda_2 x},$$

where A and B are arbitrary constants.

Case 2: $p^2 - 4q < 0$

If the discriminant is negative ($p^2 - 4q < 0$) then λ has two complex conjugate roots of the form

$$\lambda = \mu \pm i\omega \quad \text{with} \quad \mu = -\frac{1}{2}p \quad \text{and} \quad \omega = \frac{1}{2}\sqrt{4q - p^2}.$$

The general solution of the homogeneous ODE can then be written as

$$y = A e^{\mu x} \cos(\omega x) + B e^{\mu x} \sin(\omega x),$$

where A and B are arbitrary constants.

Case 3: $p^2 - 4q = 0$

If the discriminant is zero ($p^2 - 4q = 0$) then the characteristic polynomial has one double root

$$\lambda_{1,2} = \lambda = -\frac{1}{2}p$$

giving only one fundamental solution $y_1 = e^{\lambda x} = e^{-px/2}$. However another fundamental solution is $y_2 = x e^{\lambda x} = x e^{-px/2}$ (*Exercise:* check this by substitution). The general solution of the homogeneous ODE can therefore be written as

$$y = A e^{-px/2} + B x e^{-px/2},$$

where A and B are arbitrary constants.

3.2.2 The particular solution of the inhomogenous ODE: The method of undetermined coefficients

- The method of undetermined coefficients is, more or less, a process of trial and error, or guesswork, based on making a suitable initial assumption about the overall form of the solution.
- The method and its pitfalls are best illustrated with an example:

$$y'' + py' + qy = A e^{ax}.$$

Initial ansatz:

Given that the RHS e^{ax} retains its functional form when differentiated, it is tempting to try a solution in the form $y = C e^{ax}$, having $y' = C a e^{ax}$ and $y'' = C a^2 e^{ax}$, so that

$$C a^2 e^{ax} + p C a e^{ax} + q C e^{ax} = A e^{ax} \quad \text{or} \quad (a^2 + pa + q)C = A$$

which requires that $C = \frac{A}{a^2 + pa + q}$, leading to the particular solution

$$y = y_p(x) = \frac{A}{a^2 + pa + q} e^{ax} \quad \text{provided} \quad a^2 + pa + q \neq 0.$$

Modification if a is a (single) root of the characteristic polynomial

If $a^2 + pa + q = 0$ the initial ansatz, that $y = C e^{ax}$, is obviously inadequate. We note that this case arises if the a happens to be a root of the characteristic polynomial of the associated homogeneous ODE. In this case, another ansatz is appropriate. We assume, instead, that

$$y = C x e^{ax} \quad \text{so that} \quad y' = C(1 + ax) e^{ax}, \quad y'' = C(2a + a^2 x) e^{ax}.$$

In this case the ODE gives

$$C(2a + a^2x)e^{ax} + pC(1 + ax)e^{ax} + qCx e^{ax} = Ae^{ax}$$

or

$$(x \underbrace{(a^2 + ap + q)}_{=0} + 2a + p)C = (2a + p)C = A$$

since $a^2 + ap + q = 0$. Thus we find that $C = \frac{A}{2a+p}$, leading to the particular solution

$$y = y_p(x) = \frac{A}{2a+p} x e^{ax} \quad \text{provided } a^2 + pa + q = 0 \quad \text{and } 2a + p \neq 0.$$

Modification if a is a double root of the characteristic polynomial

If both $a^2 + pa + q$ and $2a + p$ are zero, then both guesses, that $y = Ce^{ax}$ or $y = Cxe^{ax}$, are obviously inadequate. We note that this case arises if a is a double root of the characteristic polynomial. In this case, yet another ansatz is appropriate. We now assume that

$$y = Cx^2 e^{ax} \quad \text{so that } y' = C(2x + ax^2)e^{ax}, \quad y'' = C(2 + 4ax + a^2x^2)e^{ax}.$$

In this case the ODE gives

$$C(2 + 4ax + a^2x^2)e^{ax} + pC(2x + ax^2)e^{ax} + qCx^2 e^{ax} = Ae^{ax}$$

or

$$(x^2 \underbrace{(a^2 + ap + q)}_{=0} + x \underbrace{2(2a + p)}_{=0} + 2)C = 2C = A$$

since $a^2 + ap + q = 0$ and $2a + p = 0$. Thus we find that $C = \frac{1}{2}A$, leading to the particular solution

$$y = y_p(x) = \frac{1}{2}Ax^2 e^{ax} \quad \text{provided } a^2 + pa + q = 0 \quad \text{and } 2a + p = 0.$$

- This example shows that a particular solution of the ODE $y'' + py' + qy = Ae^{ax}$, with constant coefficients p and q , typically takes the form $Cx^m e^{ax}$ for an integer power m that depends on whether or not e^{ax} and xe^{ax} are solutions of the homogeneous equation.
- Based on this observation we can formulate the “method of undetermined coefficients” for inhomogenous, constant-coefficient of the form

$$y'' + py' + qy = A_1 r_1(x) + A_2 r_2(x) + \cdots + A_n r_n(x)$$

where the RHS is a linear combination of n given, linearly-independent functions $r_i(x)$ ($i = 1, \dots, n$).

The idea is the following:

1. We initially try to find a particular solution that contains the same (linearly independent) functions that occur on the RHS:

$$y_P^{[initial]}(x) = C_1 r_1(x) + C_2 r_2(x) + \cdots + C_n r_n(x)$$

with undetermined (constant) coefficients C_i ($i = 1, \dots, n$). The plan is to insert this into the ODE and to collect the coefficients that multiply the same functions $r_i(x)$ ($i = 1, \dots, n$). Since the $r_i(x)$ are linearly independent, their linear combination can only vanish if the coefficients multiplying them vanish individually. This provides n equations for the n unknown coefficients C_i ($i = 1, \dots, n$). Bingo!

2. This doesn't work, however, if the derivative of any of the $r_i(x)$ cannot be expressed as a linear combination of the terms in $y_P^{[initial]}$. [In the above example, the derivatives of $r_1(x) = e^{ax}$ were simply multiples of e^{ax} , so no additional functions arose. However, if $r_1(x) = x^2$, say, the differentiation of $y_P^{[initial]}$ would also produce $r_1'(x) = 2x$ and $r_1''(x) = 2$.]

To deal with such cases, we generalise our ansatz to the form

$$\begin{aligned} y_P^{[better]}(x) &= C_1 r_1(x) + C_2 r_2(x) + \cdots + C_n r_n(x) \\ &+ D_1 r_1'(x) + D_2 r_2'(x) + \cdots + D_n r_n'(x) \\ &+ E_1 r_1''(x) + E_2 r_2''(x) + \cdots + E_n r_n''(x), \end{aligned}$$

where we set the coefficients E_i and D_i ($i = 1, \dots, n$) that multiply terms that are already contained in $y_P^{[initial]}(x)$ to zero.

3. Finally, we have to deal with the case where some of the terms in $y_P^{[better]}$ are solutions of the homogeneous ODE $y'' + p y' + q y = 0$. Let $\tilde{r}(x)$ be a term in $y_P^{[better]}(x)$ that is a solution of the homogeneous ODE. For each such term, we replace $\tilde{r}(x)$ by $x^m \tilde{r}(x)$, where m is the smallest positive integer for which $x^m \tilde{r}(x)$ does not solve the homogeneous ODE. If the derivatives of $x^m \tilde{r}(x)$ create new linearly independent functions, not yet contained in $y_P^{[better]}$, add these too.

3.3 Some nonlinear second-order ODEs

In a few cases, second-order ODEs can be solved as first-order ODEs. Two important cases are those that take the form

$$\frac{d^2 y}{dt^2} = f\left(y, \frac{dy}{dt}\right) \quad \text{or} \quad \frac{d^2 y}{dt^2} = f\left(t, \frac{dy}{dt}\right)$$

when describing $y(t)$. The first of these represents second-order ODEs that are autonomous, which is to say that they do not depend on t (apart from differentiating with respect to t). The second represents second-order ODEs that do not depend on y (except as derivatives of y).

3.3.1 Second-order ODEs for $y(t)$ that do not depend on y

Such ODEs take the form

$$y'' = f(t, y').$$

All we need to do is note that this is actually a first-order ODE for $y'(t)$. If we write, $v(t) = y'(t)$ then the ODE is clearly a first-order ODE for v , namely

$$v' = f(t, v).$$

If this is solved to find a solution $v(t)$, then $y(t)$ is a solution of the first-order ODE $y' = v(t)$.

3.3.2 Autonomous second-order ODEs

Autonomous second-order ODEs which, when describing $y(t)$ have the form

$$y'' = f(y, y')$$

can also be solved by writing $v = y'(t)$, but in a different way. Differentiating $y'(t) = v$ gives

$$y'' = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy}.$$

The ODE can therefore be rewritten in the form

$$v \frac{dv}{dy} = f(y, v)$$

which, if we think of v as being a function of y , is a first-order ODE for v . If we can solve for $v = v(y)$ then $y(t)$ is a solution of the first-order ODE $y' = v(y)$.