2.2 Solving first-order ODEs

It is not always possible to solve ordinary differential equations analytically. Even when the solution of an ODE is known to exist, it is not always possible to find the solution in terms of known standard functions, such as powers, exponentials, logarithms, trigonometric functions or even more specialised functions such as Bessel functions, error functions, etc.

Other methods, including numerical techniques, perturbation methods or graphical approaches, may need to be invoked to obtain or, if necessary, to approximate the solution. In very many cases, that cannot be solved in terms of standard functions, an infinite power series solution can be developed to provide the correct solution at least over some interval where the series converges, although more approximate solutions might sometimes be more revealing.

On the other hand, there are several forms of first-order ODE that can be solved analytically. There are also some particular ODEs which can be solved by using suitable transformations. We will now outline each of these types of equation and the ways in which they can be solved.

2.2.1 Separable first-order ODEs

- An ODE describing $y(x)$ is separable if it can be rearranged into the form

$$g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = h(x)$$

for some functions $g(\cdot)$ and $h(\cdot)$. Formally, multiplying both sides by $\mathrm{d}x$, this produces the form of the ODE

$$g(y)\,\mathrm{d}y = h(x)\,\mathrm{d}x$$

in which all dependence on x and $\mathrm{d}x$ has been separated onto one side of the equation with all dependence on y and $\mathrm{d}y$ on the other. In this form, each side of the equation can simply be integrated to provide a solution

$$\int g(y)\,\mathrm{d}y = \int h(x)\,\mathrm{d}x + A$$

for an arbitrary constant of integration A. It might not always be possible to solve the resulting formula explicitly for y, but, if that is the case, the formula does at least provide an implicit relationship between x and the solution y.

Example 1. The ODE $(1-y^2)\sin(x)y' - y\cos(x) = 0$ can be rearranged, through dividing by y and by $\sin(x)$, to give

$$\Big(\frac{1}{y} - y\Big)y' = \frac{\cos(x)}{\sin(x)}.$$

Integrating both sides therefore gives

$$\int \Big(\frac{1}{y} - y\Big)\mathrm{d}y = \int \frac{\cos(x)}{\sin(x)}\,\mathrm{d}x + A$$

so that

$$\ln|y| - \tfrac{1}{2}y^2 = \ln|\sin x| + A.$$

This result cannot be solved explicitly for the solution $y(x)$.

Example 2. The function $v(z)$ satisfies $\frac{\mathrm{d}v}{\mathrm{d}z} = -3z^2\,\mathrm{e}^v$ with $v(0) = 0$. Rearranging the ODE gives

$$\mathrm{e}^{-v}\frac{\mathrm{d}v}{\mathrm{d}z} = -3z^2$$

so that

$$\int \mathrm{e}^{-v}\,\mathrm{d}v = -3\int z^2\,\mathrm{d}z + A \quad \text{or} \quad -\mathrm{e}^{-v} = -z^3 + A$$

giving the solution

$$v(z) = -\ln(z^3 - A)$$

for an arbitrary constant A. Applying the initial condition $v(0) = 0$ requires

$$0 = -\ln(-A) \quad \text{so that} \quad A = -1.$$

The unique solution that satisfies the initial condition is therefore

$$v = -\ln(z^3 + 1).$$

2.2.2 First-order ODEs of homogeneous type

- An ODE for $y(x)$ is of homogeneous type if it can be rearranged into the form

$$y' = f(y/x)$$

and it can be simplified by using the substitution

$$z(x) = y(x)/x \quad \text{or} \quad y(x) = x\ z(x)$$

where $z(x)$ is another function of x. Differentiating gives

$$y' = xz' + z \quad \text{so that} \quad xz' + z = f(z) \quad \text{or} \quad z' = \frac{f(z) - z}{x}$$

which is a separable equation, that we already know how to solve for $z(x)$.

  - The transformation $z = y/x$ reduces an ODE of homogeneous type describing $y(x)$ to a separable ODE describing $z(x)$.
  - Once a solution for $z(x)$ is found, the solution for $y(x)$ is simply $y = x\ z(x)$.

Example. The ODE for $u(t)$

$$tu\frac{\mathrm{d}u}{\mathrm{d}t} = u^2 + 3t\sqrt{tu} \qquad \text{becomes} \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{u}{t} + 3\sqrt{t/u}$$

after dividing both sides by tu. The right hand side is a function of u/t so that the ODE is of homogeneous type. It can be turned into a separable ODE using the substitution $z(t) = u/t$ or

$$u = tz \quad \text{so that} \quad u' = tz' + z \quad \text{giving} \quad tz' + z = z + 3z^{-1/2}$$

Simplifying and separating variables gives

$$z' = 3z^{-1/2}/t \quad \text{or} \quad z^{1/2}z' = 3/t$$

and so integrating gives

$$\int z^{1/2}\,\mathrm{d}z = 3\int \frac{\mathrm{d}t}{t} + A \quad \text{or} \quad \tfrac{2}{3}z^{3/2} = 3\ln|t| + A$$

for an arbitrary constant A. Solving for z and substituting into $u = tz$ to find the solution u gives

$$z = \left(\tfrac{9}{2}\ln|t| + B\right)^{2/3} \quad \text{and so} \quad u = tz = t\left(\tfrac{9}{2}\ln|t| + B\right)^{2/3}$$

where B is an arbitrary constant.

Note. The constant $\frac{9}{2}A$ is arbitrary, so there is no need to write it as $\frac{9}{2}A$ and we might as well write it as one single symbol B.
There is no error made if we leave it as $\frac{9}{2}A$, but writing it as B is a little simpler.

2.2.3 Linear first-order ODEs

- Linear ODEs have the form
$$a(x)\frac{dy}{dx} + b(x)y = c(x),$$
where $a(x), b(x)$ and $c(x)$ are given functions.

- We divide the ODE by $a(x)$ to transform it into its *standard form*
$$\frac{dy}{dx} + p(x)y = q(x). \tag{1}$$

- This equation can be integrated by using the *integrating factor*
$$I(x) = e^{\int p(x)dx}.$$
Note that the constant of integration can be discarded when determining the integrating factor.

- After multiplying the standard form of the ODE (1) by the integrating factor, it can be rewritten in the form
$$\frac{d}{dx}\left(y\ I(x)\right) = q(x)I(x).$$

- This can easily be integrated to give
$$y(x) = \frac{1}{I(x)}\int q(x)\ I(x)\ dx$$

- This integration produces one constant of integration which has to be determined from the initial condition.

- Table 1 shows a step-by-step illustration of how this method works.

Table 1: The integration of a linear first-order ODE.

Step	General Procedure	Example
1. Identify the terms:	$a(x)\frac{dy}{dx} + b(x)y = c(x)$	$\underbrace{2x}_{a(x)}\frac{dy}{dx} + \underbrace{4x^2}_{b(x)}y = \underbrace{2x^2}_{c(x)}$
2. Transform into the standard form (i.e. divide by $a(x)$ if required):	$\frac{dy}{dx} + p(x)y = q(x)$	$\frac{dy}{dx} + \underbrace{2x}_{p(x)}y = \underbrace{x}_{q(x)}$
3. Determine the integrating factor (ignore the constant of integration):	$I(x) = e^{\int p(x)dx}$	$I(x) = e^{\int 2x dx} = e^{x^2}$
4. Multiply the ODE by $I(x)$ and rewrite the LHS:	$\frac{d}{dx}\left(y\ I(x)\right) = q(x)I(x)$	$\frac{d}{dx}\left(y\ e^{x^2}\right) = xe^{x^2}$
5. Integrate (don't forget the constant of integration!):	$y\ I(x) = \int q(x)I(x)dx$	$y\ e^{x^2} = \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + C$
6. Solve for $y(x)$:	$y(x) = \frac{1}{I(x)}\int q(x)I(x)dx$	$y(x) = \frac{1}{2} + Ce^{-x^2}$

- Again, the last step produces the *general solution.* The *specific solution* is determined by fixing the constant via the initial condition.

- **Note:** The initial condition cannot be placed where either $p(x)$ or $q(x)$ are singular. If the ODE is given in its general form, $a(x)y' + b(x)y = c(x)$, this situation arises at points where the coefficient multiplying the highest derivative, $a(x)$, vanishes. This is nearly always a "sign of trouble" and we will encounter other examples throughout this course.

- The solution of a linear first-order ODEs can be written as

  $$y(x) = y_P(x) + y_H(x)$$

  where $y_P(x)$ is *any* particular solution of the ODE $y' + p(x)\,y = q(x)$, and $y_H(x)$ is the *general* solution of the corresponding *homogenous ODE*[2] $y' + p(x)\,y = 0$. This is a generic feature of all linear ODEs, not just linear ODEs of first order.

[2]Do not confuse the "ODE of homogenous type", discussed in section 2.2.2, with the *homogenous linear ODE*!