MATH10222: SOLUTIONS[1] VI

1. **An algebraic example for perturbation methods: Roots of polynomials**

(a) Inserting the expansion

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... \tag{1}$$

into the polynomial

$$x^4 + \epsilon x - 1 = 0 \tag{2}$$

yields

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...)^4 + \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...) - 1 = 0. \tag{3}$$

The lowest-order terms (in ϵ) in the expansion of x^4 are

$$(x_0+\epsilon x_1+\epsilon^2 x_2+...)^4 = (x_0)^4+4\,(x_0)^3\,(\epsilon x_1)+4\,(x_0)^3\,(\epsilon^2 x_2)+6\,(x_0)^2\,(\epsilon x_1)^2+\mathcal{O}(\epsilon^3),$$

where we have used brackets to indicate where the various terms come from. Now collect like powers of ϵ in (3)

$$\left(x_0^4 - 1\right) + \epsilon\left(4x_0^3 x_1 + x_0\right) + \epsilon^2\left(4x_0^3 x_2 + 6x_0^2 x_1^2 + x_1\right) + ... = 0.$$

Setting the coefficients multiplying the powers of ϵ to zero then yields the following sequence of equations:

i. The "leading-order" equation, associated with the terms multiplied by ϵ^0, is

$$x_0^4 - 1 = 0 \quad\Longrightarrow\quad x_0 = \pm 1, \pm i.$$

Following the suggestion in the question, we only consider the solution $x_0 = 1$ in the subsequent analysis.

ii. The next equation, associated with the terms multiplied by ϵ^1, is

$$4x_0^3 x_1 + x_0 = 0.$$

Using the value for $x_0 = 1$, computed above, this yields

$$4x_1 + 1 = 0 \quad\Longrightarrow\quad x_1 = -\frac{1}{4}.$$

iii. Next, we have the terms that are multiplied by ϵ^2,

$$4x_0^3 x_2 + 6x_0^2 x_1^2 + x_1 = 0.$$

Again, we insert the previously computed coefficients to obtain

$$4x_2 + 6\,\frac{1}{16} - \frac{1}{4} = 0 \quad\Longrightarrow\quad x_2 = -\frac{1}{32}.$$

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Hence, an approximation for the root "near" $x = 1$ is given by

$$x = 1 - \frac{1}{4}\,\epsilon - \frac{1}{32}\,\epsilon^2 + ... \qquad (4)$$

(b) Inserting $\epsilon = 0.2$ into the three-term approximation $x = 1 - 1/4\,\epsilon - 1/32\,\epsilon^2$ yields $x = 0.94875$ which shares the first five digits with the "exact" solution 0.9487561315 (obtained numerically). Rounding the exact solution to 5 digits would give 0.94876 so in that sense there's a discrepancy at that level – you can now argue forever if this means four- or five-digit accuracy. It's pretty damn good by any standard! In fact, we're benefiting slightly from the fact that the next term in the expansion happens to be zero, $x_3 = 0$, so the error is actually smaller than suggested by the truncated expansion (4) because

$$x = 1 - \frac{1}{4}\,\epsilon - \frac{1}{32}\,\epsilon^2 + \mathcal{O}(\epsilon^4). \qquad (5)$$

(c) The roots of the quadratic polynomial

$$\epsilon x^2 + x - 1 = 0 \qquad (6)$$

are given by

$$x_{[1,2]} = -\frac{1}{2\epsilon}\left(1 \pm \sqrt{1+4\epsilon}\right).$$

Using the expansion

$$\sqrt{1+4\epsilon} = 1 + 2\epsilon - 2\epsilon^2 + 4\epsilon^3 + ...$$

(convergent for $|\epsilon| < 1/4$) shows that

$$\begin{aligned} x_{[1]} &= -\frac{1}{2\epsilon}\left(1 - \sqrt{1+4\epsilon}\right) \\ &= -\frac{1}{2\epsilon}\left(-2\epsilon + 2\epsilon^2 - 4\epsilon^3 + ...\right) \\ &= 1 - \epsilon + 2\epsilon^2 + ... \end{aligned}$$

The behaviour of this root can therefore be captured by an expansion in positive powers of ϵ. Furthermore, we see that

$$x_{[1]} \to 1 \quad \text{as } \epsilon \to 0,$$

which is obviously the solution of the polynomial $x - 1 = 0$, obtained by setting $\epsilon = 0$ in (6).

The second root is not so well-behaved:

$$\begin{aligned} x_{[2]} &= -\frac{1}{2\epsilon}\left(1 + \sqrt{1+4\epsilon}\right) \\ &= -\frac{1}{2\epsilon}\left(2 + 2\epsilon - 2\epsilon^2 + 4\epsilon^3 + ...\right) \\ &= -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + ... \end{aligned}$$

Clearly, this behaviour cannot be captured by an expansion in positive powers of ϵ. To analyse the origin of this difficulty, we note that

$$x_{[2]} \to -\infty \quad \text{as } \epsilon \to 0,$$

so the second root "escapes to infinity (and beyond?)" as $\epsilon \to 0$. This shouldn't be entirely unexpected since in the limit of $\epsilon \to 0$ a key feature of the problem is lost: We're changing a second-order polynomial (which has two roots) to a first-order polynomial (which has only one root). The structure of the solution for $\epsilon = 0$ is therefore fundamentally different from that for small (but finite) values of ϵ, no matter how small ϵ is.

This is another example that illustrates that special care tends to be required in situations in which the "highest-order term" (the highest power in a polynomial, the highest derivative in an ODE, ...) vanishes. We stress that perturbation methods *can* be adjusted to successfully deal with such "singular" (as opposed to "regular") perturbation problems. However, this is beyond the scope of this lecture course which is only supposed to give you an introduction to the main ideas behind the method. If the method appeals to you (and I hope it does!), you can learn a lot more about it in your second and third year.

Incidentally, here's the exact, closed-form solution for (just one of) the four roots. You won't be surprised to hear that I used MAPLE to calculate it. A nice illustration that closed-form solutions are not necessarily very useful in practice. How long would it take you to evaluate this expression for any specific value of ϵ?

$$\frac{1}{12}\sqrt{6}\sqrt{\frac{\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(2/3)}-48}{\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(1/3)}}} + \frac{1}{12}\left(-\left(6\sqrt{\frac{\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(2/3)}-48}{\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(1/3)}}}\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(2/3)} - 288\left(\frac{\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(2/3)}-48}{\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(1/3)}}\right)^{\frac{1}{2}} + 72\,\varepsilon\sqrt{6}\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(1/3)}\right)\Bigg/\left(\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(1/3)}\left(\frac{\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(2/3)}-48}{\left(108\,\varepsilon^2+12\sqrt{768+81\,\varepsilon^4}\right)^{(1/3)}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{2}}$$

2. **Perturbation methods for linear ODEs: A mechanical oscillator with a weak spring**

   (a) Inserting the perturbation expansion

   $$x(t) = x_0(t) + \epsilon\, x_1(t) + \epsilon^2\, x_2(t) + ...,$$

   into the ODE

   $$\ddot{x} + \dot{x} + \epsilon x = 0$$

   yields

   $$\left(\ddot{x_0}(t) + \epsilon\, \ddot{x_1}(t) + \epsilon^2\, \ddot{x_2}(t) + ...\right) + \left(\dot{x_0}(t) + \epsilon\, \dot{x_1}(t) + \epsilon^2\, \dot{x_2}(t) + ...\right) +$$

   $$+\epsilon\left(x_0(t) + \epsilon\, x_1(t) + \epsilon^2\, x_2(t) + ...\right) = 0.$$

   We use the same expansion in the initial conditions:

   $$x(t=0) = x_0(t=0) + \epsilon\, x_1(t=0) + \epsilon^2\, x_2(t=0) + ... = 1$$

   and

   $$\dot{x}(t=0) = \dot{x_0}(t=0) + \epsilon\, \dot{x_1}(t=0) + \epsilon^2\, \dot{x_2}(t=0) + ... = 0.$$

   Collecting like powers of ϵ now generates the following sequence of initial value problems:

   i. The "leading-order" problem is the problem that we would have obtained by setting $\epsilon = 0$ in the original ODE:

   $$\ddot{x_0} + \dot{x_0} = 0,$$

   subject to

   $$x_0(t=0) = 1 \quad \text{and} \quad \dot{x_0}(t=0) = 0.$$

   Your superb knowledge of second-order constant-coefficient ODEs should enable you to confirm immediately that the solution of this IVP is given by

   $$x_0(t) = 1.$$

   [Note that this makes perfect sense in the context of the mechanical oscillator, because if there's no spring ($\epsilon = 0$), the mass will simply stay in its initial position. If that's not obvious to you, recall that the force that the damper exerts on the mass is proportional to its velocity. Since the initial condition requires the mass to be at rest initially, it'll stay at rest (and hence at its original position) indefinitely.]

   ii. The next problem, given by the terms that are multiplied by ϵ, is

   $$\ddot{x_1} + \dot{x_1} = -x_0(t) = -1, \tag{7}$$

   subject to

   $$x_1(t=0) = 0 \quad \text{and} \quad \dot{x_1}(t=0) = 0. \tag{8}$$

The solution of the homogeneous ODE is given by

$$x_{1[H]}(t) = A + B\,e^{-t} \tag{9}$$

for arbitrary constants A and B. This shows that the RHS of (7) is a solution of the homogeneous ODE. Therefore we try the ansatz $x_{1[P]}(t) = C\,t$ for the particular solution. Inserting this into (7) shows that $C = -1$. Finally, applying the initial conditions (8) to the general solution $x_1(t) = x_{1[H]}(t) + x_{1[P]}(t)$ yields $A = 1$ and $B = -1$, so

$$x_1(t) = 1 - t - e^{-t}.$$

iii. The next problem, given by the terms that are multiplied by ϵ^2, is [2]

$$\ddot{x_2} + \dot{x_2} = -x_1(t), \tag{10}$$

subject to

$$x_2(t=0) = 0 \quad \text{and} \quad \dot{x_2}(t=0) = 0. \tag{11}$$

Inserting $x_1(t)$ from the previous problem yields

$$\ddot{x_2} + \dot{x_2} = t + e^{-t} - 1. \tag{12}$$

The solution of the homogeneous ODE is the same as before, i.e. $x_{2[H]}(t) = A + B\,e^{-t}$ for (different!) arbitrary constants A and B. We note that the constant term and the e^{-t} term on the RHS of (12) are solutions of the homogeneous ODE. We therefore try the ansatz

$$\begin{aligned} x_{2[P]}(t) &= D\,t\,e^{-t} + E\,t^2 + F\,t \\ \dot{x}_{2[P]}(t) &= D\,(1-t)\,e^{-t} + 2E\,t + F \\ \ddot{x}_{2[P]}(t) &= D\,(t-2)\,e^{-t} + 2E \end{aligned}$$

for the particular solution. Inserting these into (12) yields

$$\Big(D\,(t-2)\,e^{-t} + 2E\Big) + \Big(D\,(1-t)\,e^{-t} + 2E\,t + F\Big) = t + e^{-t} - 1,$$

so

$$\Big(2E + F + 1\Big) + \Big(2E - 1\Big)\,t + \Big(-2D + D - 1\Big)e^{-t} + \Big(D - D\Big)\,t\,e^{-t} = 0.$$

This requires $E = 1/2$, $D = -1$ and $F = -2$, so

$$x_{2[P]}(t) = -t\,e^{-t} + \frac{1}{2}\,t^2 - 2\,t.$$

Finally, applying the initial conditions (11) to the general solution $x_2(t) = x_{2[H]}(t) + x_{2[P]}(t)$ yields $A = 3$ and $B = -3$, so

$$x_2(t) = 3 - 3e^{-t} - t\,e^{-t} + \frac{1}{2}\,t^2 - 2\,t.$$

[2]In fact, it is easy to see that all subsequent problems have the same structure: They are given by

$$\ddot{x_i} + \dot{x_i} = -x_{i-1}(t),$$

subject to

$$x_i(t=0) = 0 \quad \text{and} \quad \dot{x_i}(t=0) = 0,$$

where $i = 1, 2, 3, ...$

(b) Here are some plots of exact and approximate solutions for $\epsilon = 0.2$ (the higher-order corrections not derived above were obtained by continuing the perturbation scheme with MAPLE – a powerful symbolic maths package).

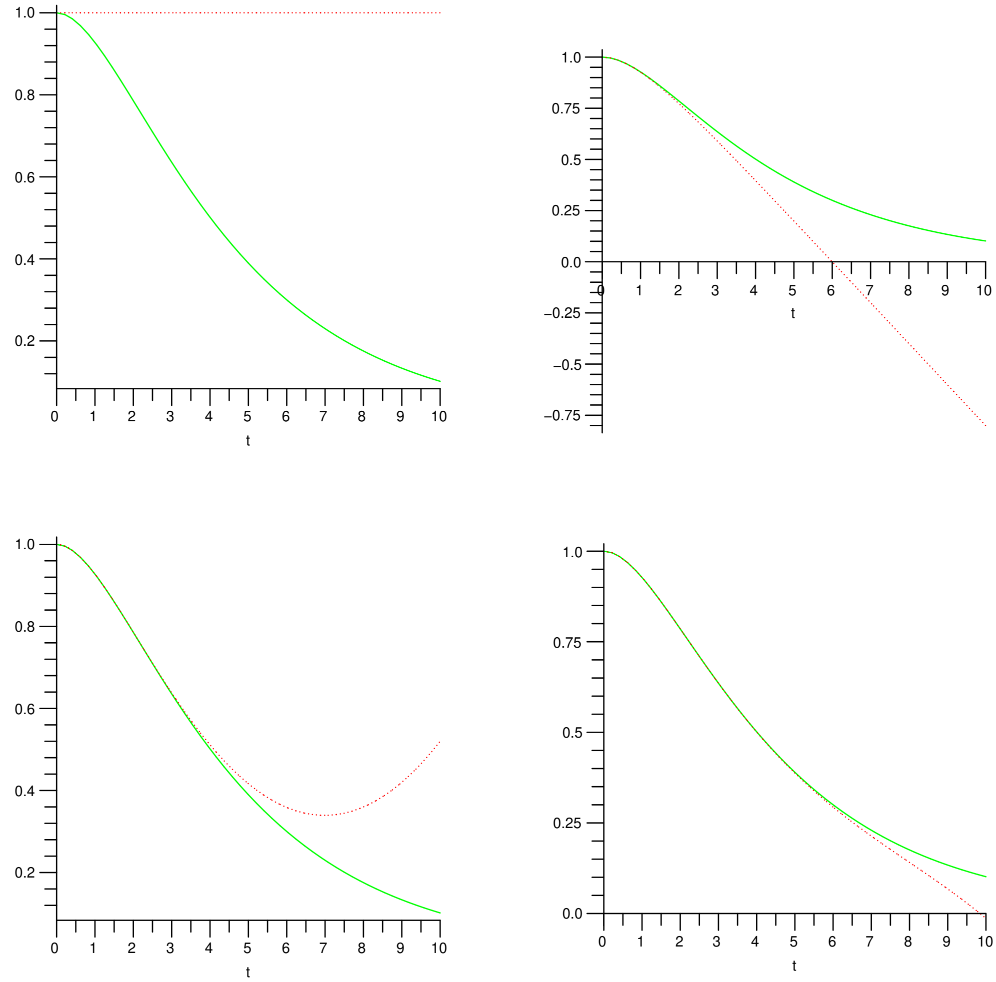

Figure 1: Exact solution (solid line) and one-, two-, three- and four-term approximate solutions (from top left to bottom right; dashed) for $\epsilon = 0.2$.

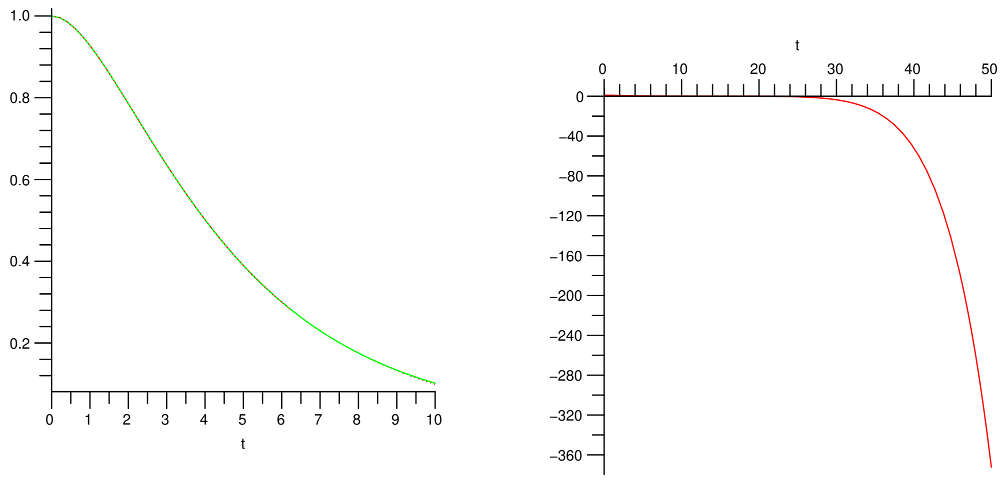

Figure 2: The five-term expansion (dashed) is graphically indistinguishable from the exact solution (solid line) until $t \approx 6$; see left figure. However, for sufficiently large values of t all approximate solutions ultimately diverge. This is illustrated by the plot (over a larger range of t values) on the right.

3. **Perturbation methods for non-linear ODEs: Getting rid of dead cats**

Following the hint in the question, we first expand the term $(1+\epsilon x)^{-2}$ in a binomial series

$$(1+\epsilon x)^{-2} = 1 - 2(\epsilon x) + 3(\epsilon x)^2 - ...$$

This transforms the ODE $\ddot{x} + (1+\epsilon x)^{-2} = 0$ into

$$\ddot{x} + 1 - 2(\epsilon x) + 3(\epsilon x)^2 - ... = 0.$$

Inserting the perturbation expansion

$$x(t) = x_0(t) + \epsilon\, x_1(t) + \epsilon^2\, x_2(t) + ...,$$

then yields

$$\Big(\ddot{x}_0+\epsilon\,\ddot{x}_1+\epsilon^2\,\ddot{x}_2+...\Big)+1-2\epsilon\Big(x_0+\epsilon\,x_1+\epsilon^2\,x_2+...\Big)+3\epsilon^2\Big(x_0+\epsilon\,x_1+\epsilon^2\,x_2+...\Big)^2-... = 0,$$

or, collecting powers of ϵ,

$$\Big(\ddot{x}_0 + 1\Big) + \epsilon\,\Big(\ddot{x}_1 - 2x_0\Big) + \epsilon^2\,\Big(\ddot{x}_2 - 2x_1 + 3x_0^2\Big) + ... = 0.$$

We use the same expansion in the initial conditions

$$x(t=0) = x_0(t=0) + \epsilon\, x_1(t=0) + \epsilon^2\, x_2(t=0) + ... = 0$$

and

$$\dot{x}(t=0) = \dot{x}_0(t=0) + \epsilon\, \dot{x}_1(t=0) + \epsilon^2\, \dot{x}_2(t=0) + ... = 1.$$

Collecting like powers of ϵ then generates the following sequence of initial value problems:

(a) As always, the "leading-order" problem is the problem that we would have obtained by setting $\epsilon = 0$ in the original ODE:

$$\ddot{x}_0 = -1, \tag{13}$$

subject to

$$x_0(t=0) = 0 \quad \text{and} \quad \dot{x}_0(t=0) = 1. \tag{14}$$

Integrating (13) twice and applying the initial conditions (14) yields

$$x_0(t) = t - \frac{1}{2}\,t^2.$$

(b) At next order we have

$$\ddot{x}_1 = 2x_0(t) = 2t - t^2. \tag{15}$$

subject to

$$x_1(t=0) = 0 \quad \text{and} \quad \dot{x}_1(t=0) = 0. \tag{16}$$

This is again most easily solved by integrating the ODE (15) twice and applying the initial conditions (16). The result is

$$x_1(t) = \frac{1}{3}\,t^3 - \frac{1}{12}\,t^4.$$

(c) Collecting terms that are multiplied by ϵ^2 gives:

$$\ddot{x}_2 = 2x_1 - 3x_0^2 = \frac{2}{3}\,t^3 - \frac{1}{6}\,t^4 - 3\left(t^2 - t^3 + \frac{1}{4}\,t^4\right),$$

i.e.

$$\ddot{x}_2 = -3t^2 + \frac{11}{3}\,t^3 - \frac{11}{12}\,t^4, \tag{17}$$

subject to

$$x_1(t=0) = 0 \quad \text{and} \quad \dot{x}_1(t=0) = 0. \tag{18}$$

Again we integrate the ODE (17) twice and apply the initial conditions (18) to obtain

$$x_2(t) = -\frac{1}{4}\,t^4 + \frac{11}{60}\,t^5 - \frac{11}{360}\,t^6.$$

If you like integrating powers of t (and who doesn't!), you can keep going for as long as you like – the maths doesn't get any harder in subsequent problems. However, I suggest we stop here and record

$$x(t) = t - \frac{1}{2}\,t^2 + \epsilon\left(\frac{1}{3}\,t^3 - \frac{1}{12}\,t^4\right) + \epsilon^2\left(-\frac{1}{4}\,t^4 + \frac{11}{60}\,t^5 - \frac{11}{360}\,t^6\right) + \mathcal{O}(\epsilon^3)$$

as our final result.

So what does all this mean for our stockbroker? Will he manage to eject his dead cat from this planet? Hmmm, I'm afraid I have bad news for you (or him): The case $\epsilon \ll 1$ considered here corresponds to the limit in which the cat's initial upward velocity is relatively small. Therefore, the analysis is only applicable if the cat is thrown high enough for variations in the earth's gravitational field to matter but not by very much... $\Longrightarrow$ Ask Rich Hewitt to tell you about the "escape problem" in his part of the course (he probably will anyway).